Engineering Reading Room

UNIVERSITY OF ILLINOIS BULLETIN

ISSUED WEEKLY

Vol. XXVII July 1, 1930 No. 44

[Entered as second-class matter December 11, 1912, at the post office at Urbana, Illinois, under the Act of August 24, 1912. Acceptance for mailing at the special rate of postage provided for in section 1103, Act of October 3, 1917, authorized July 31, 1918.]

THE TORSIONAL EFFECT OF TRANSVERSE BENDING LOADS ON CHANNEL BEAMS

BY

FRED B. SEELY

WILLIAM J. PUTNAM

WILLIAM L. SCHWALBE

D620.6
I16

BULLETIN No. 211

ENGINEERING EXPERIMENT STATION

PUBLISHED BY THE UNIVERSITY OF ILLINOIS, URBANA

PRICE: THIRTY-FIVE CENTS

THE Engineering Experiment Station was established by act of the Board of Trustees of the University of Illinois on December 8, 1903. It is the purpose of the Station to conduct investigations and make studies of importance to the engineering, manufacturing, railway, mining, and other industrial interests of the State.

The management of the Engineering Experiment Station is vested in an Executive Staff composed of the Director and his Assistant, the Heads of the several Departments in the College of Engineering, and the Professor of Industrial Chemistry. This Staff is responsible for the establishment of general policies governing the work of the Station, including the approval of material for publication. All members of the teaching staff of the College are encouraged to engage in scientific research, either directly or in coöperation with the Research Corps composed of full-time research assistants, research graduate assistants, and special investigators.

To render the results of its scientific investigations available to the public, the Engineering Experiment Station publishes and distributes a series of bulletins. Occasionally it publishes circulars of timely interest, presenting information of importance, compiled from various sources which may not readily be accessible to the clientele of the Station.

The volume and number at the top of the front cover page are merely arbitrary numbers and refer to the general publications of the University. *Either above the title or below the seal* is given the number of the Engineering Experiment Station bulletin or circular which should be used in referring to these publications.

For copies of bulletins or circulars or for other information address

THE ENGINEERING EXPERIMENT STATION,
UNIVERSITY OF ILLINOIS,
URBANA, ILLINOIS

UNIVERSITY OF ILLINOIS
ENGINEERING EXPERIMENT STATION

BULLETIN NO. 211 JULY, 1930

THE TORSIONAL EFFECT OF TRANSVERSE BENDING LOADS ON CHANNEL BEAMS

BY

FRED B. SEELY
PROFESSOR OF THEORETICAL AND APPLIED MECHANICS

WILLIAM J. PUTNAM
ASSISTANT PROFESSOR OF THEORETICAL AND APPLIED MECHANICS

WILLIAM L. SCHWALBE
ASSOCIATE IN THEORETICAL AND APPLIED MECHANICS

ENGINEERING EXPERIMENT STATION
PUBLISHED BY THE UNIVERSITY OF ILLINOIS, URBANA

5500 430 8127

UNIVERSITY
OF ILLINOIS
:: PRESS ::

CONTENTS

PAGE

I. Introduction 7
1. Purpose of Investigation 7
2. Previous Work 8
3. Scope of Tests and General Method of Testing. . 9
4. Acknowledgment 9

II. Shear Center for Channel Sections 10
5. Approximate Location of Shear Center 10
6. Experimental Determination of Shear Center . . 14
7. Observed Values of Longitudinal Stress in Channel When Loaded through Shear Center . . . 23

III. Longitudinal Stress in Channel When the Transverse Loads Produce Twisting with the Bending 25
8. Influence of Twisting on Longitudinal Stresses. . 25
9. Observed Values of Longitudinal Stress in Channel When Load Did Not Pass through Shear Center 30
10. Simplified Approximate Analysis of Longitudinal Stresses. 34
11. Additional Longitudinal Stresses 37
12. Significance of Effect of Twisting of Channel Beams 39

IV. Conclusion 40
13. Summary 40

Appendix. Bars of Narrow Cross-section Subjected to Torsion with Restraint

I. Introduction 43
1. Purpose of Appendix 43
2. Types of Sections 43
3. Loads and Restraints 43
4. Assumptions. 43
5. Method of Analysis 43

II. Transverse Bending without Torsion 45
6. The Shear Center in General 45
7. The Shear Center for Channel Sections 46

III. TORSION WITHOUT RESTRAINT 49
8. Torsion Formula 49

IV. TORSION WITH RESTRAINT 50
9. Torsion with Restraint Equivalent to Bending Plus Torsion without Restraint 50
10. Center of Rotation for an Angle Section . . . 51
11. Center of Rotation for a Channel Section . . . 52
12. Longitudinal Stress Formula 53
13. Bending Shear Couple 55
14. Differential Equation for Torsion with Restraint . 56
15. Variation of Torsional Couple and Bending Shear Couple along Length of Bar. 58
16. Bending Moment M' in the Half Section . . . 59
17. Computation of Longitudinal Stresses in a 6-in. 15.3-lb. Ship Channel. 60

LIST OF FIGURES

NO.		PAGE
1.	Stress in Channel Loaded Through Shear Center	13
2.	Sketch Showing Arrangement of Beam for Testing	14
3.	View of Channel with Load Applied through Shear Center at End of Cantilever	15
4.	View of Channel with Load Applied Approximately through the Centroid of End Section	16
5.	Curves Showing Relation between Angle of Twist and Distance from Fixed End of Cantilever	18
6.	Location of Shear Center Determined from Test Data	20
7.	Observed Longitudinal Stress in Channel Loaded through Shear Center	24
8.	General Torsional Effect of Transverse Load on Channel Beam	26
9.	Observed Longitudinal Stress in Channel Loaded Approximately through Centroid	28
10.	Observed Longitudinal Stress in Channel Loaded Back of Shear Center	31
11.	Approximate Analysis of Longitudinal Stress in Channel When Load Is Not Applied through Shear Center	35
12.	Types of Sections	44
13.	Channel Beam Subjected to Torsion with Restraint	44
14.	Section of Channel Loaded through Shear Center	46
15.	Equilibrium of Element of Channel	47
16.	Rotation of Channel Subjected to Torsion	51
17.	Rotation of Angle Subjected to Torsion	52
18.	Center of Rotation for Channel Section	53
19.	Section of Beam Subjected to Unsymmetrical Bending	54
20.	Circle of Inertia	55
21.	Circle of Inertia Applied to Unsymmetrical Bending	55
22.	Bending Shear Couple	56
23.	Channel Beam with Load Not through Shear Center	59
24.	Graphical Determination of Section Factors for a Ship Channel	61

LIST OF TABLES

1.	Experimental and Calculated Values of the Distance of Shear Center from Center Line of Web	22
2.	Values of a	37
3.	Values of Additional Longitudinal Stress at B Due to Load Applied through Centroid of Channel Cross-section	38
4.	Values of Additional Longitudinal Stress at A Due to Load Applied Back of Shear Center of Channel Cross-section	39
5.	Values of J for Various Channel Sections	50
6.	Longitudinal Stresses in 6-in. 15.3-lb. Ship Channel	63

THE TORSIONAL EFFECT OF TRANSVERSE BENDING LOADS ON CHANNEL BEAMS

I. Introduction

1. *Purpose of Investigation.*—The shapes of steel rolled I beams and channels have been designed with the main purpose of giving large resistance to bending in accordance with the simple flexure theory for beams; the torsional strength and stiffness of such sections are known to be relatively small. In order to develop the bending resistance of I beams and channels without permitting them to twist, it has been very generally assumed that transverse bending loads on these sections should be applied through the centroids of the transverse cross-sections and parallel to the webs. In the case of I beams this assumption is correct, but when a channel is so loaded it twists appreciably as it bends.

In order to cause a channel beam to bend without twisting, and to develop stresses in accordance with the simple flexure formula for beams, the loads must be applied in a plane parallel to the web but at a considerable distance back of the back of the channel. The intersection of this plane of loading with the neutral surface is called the "axis of bending"; and the intersection of this axis with a transverse cross-section of the beam is called the "shear center" for the section.

When I beams and channels are free from lateral restraint, and hence are free to twist, and are loaded so that the transverse bending loads do not pass through the axis of bending, then these members will twist as they bend. This twisting of the beam will produce large additional longitudinal stresses, provided that the beam has a transverse cross-section that remains plane and hence does not warp as the beam twists, such as the center section of a simple beam subjected to symmetrical loading, or the fixed end of a cantilever beam.

Only channel beams were tested in the investigation herein reported. In some of the channels tested the additional longitudinal stress due to the twisting caused by a load applied through the centroid of the section was more than 50 per cent of the stress that the same load developed when applied through the shear center. The maximum additional stress occurred at the section that was completely restrained from warping.

It is evident, therefore, that in the use of steel rolled channels as beams it may be important to determine where the loads should be

applied to produce bending without twisting, and also to determine the longitudinal stress in a channel beam if the channel is loaded so that twisting accompanies the bending. It is the purpose of this bulletin to present the results of tests and of analysis that will help toward a solution of these problems.

2. *Previous Work.*—In 1909-10 Prof. C. Bach* published the results of bending tests of a 30 cm. (11.8 in.) steel rolled channel used as a simple beam loaded with two equal concentrated loads, at the third points. The loads were applied through the centroids of the cross-sections. He measured the strains at the center of the beam along each of the four edges, and found that the strain along one edge of the top was much greater than that along the other edge of the top, and that a similar condition existed on the bottom edges. The experimental results thus indicated that the usual flexure formula gave values of stress in a channel beam largely in error when the channel was loaded according to the conditions that had been assumed to make the flexure formula applicable. Professor Bach also applied the loads through the center of the web and found that the channel acted more nearly in accordance with the flexure formula, but nevertheless the discrepencies between the stresses found from the measured strains and the values given by the flexure formula were appreciable. Professor Bach did not offer an explanation of the inconsistency between the test results and the assumed theory.

These tests, however, served to direct attention in a practical way to the difference in the behavior of a channel in bending under transverse loads acting through the centroids of the cross-sections from that of an I beam similarly loaded, but no satisfactory analysis of the problem appears to have been given until about ten years later.

In 1920 H. Schwyzer presented a thesis at the University of Zurich in which the shear center was determined by mathematical analysis. In 1920-21 R. Maillart,† A. Eggenschwyler,‡ and H. Zimmerman§ by somewhat different analyses brought to the attention of engineers the location and significance of the shear center for channels and some other thin sections. These investigators also obtained mathematical expressions for the additional longitudinal stress caused by the twisting of a channel when the transverse loads on the channel do not pass through the axis of bending, and they found that the experimental

*Zeit. d. Vereins deutscher Ingenieure, 1909, p. 1790, 1910, p. 382.
†Schweiz, Bauz., Vol. 77, 1921, p. 195.
‡Schweiz, Bauz., Vol. 76, 1920, p. 266.
§Bauingenieur, Vol. 2, 1921, p. 202.

results of Bach were in substantial agreement with the results of their analyses.

In 1925 Föppl and Huber* tested a 10.3-in. channel as a simple beam on a span of 20.8 ft. with a concentrated load of 5500 lb. at the middle. They also made a test of the same beam with two concentrated loads at the third points. They measured angles of twist and longitudinal strains at various sections for different lateral positions of the load. Only one beam was tested but the results of the tests agreed well with the calculated position of the shear center, and with the values of the stresses given by the mathematical analysis of the stresses in the channel; the mathematical analysis used was a modification of Timoshenko's analysis,† made in 1910, of the stresses in an I beam subjected to transverse loads that cause twisting.

3. *Scope of Tests and General Method of Testing.*—The channels tested in the investigation herein reported were the following:

4 in.– 5.3 lb.	6 in.–15.2 lb. (heavy section)
6 in.– 8.2 lb.	10 in.–15.3 lb.
6 in.–15.3 lb. (ship channel)	15 in.–33.9 lb.

These channels offer a rather wide range in values of the ratio of flange area to web area and in relative dimensions.

Each channel was tested as a horizontal cantilever beam with a vertical load applied at the end. The load consisted of a weight hung on the free end of the beam, the other end being restrained so that the section remained practically plane.

For each channel the angle of twist was measured at six or seven sections along the length of the beam for each of several lateral positions of the load, the load being applied at two or three points on each side of the back of the channel (see Figs. 2, 3, and 4 for general arrangement).

From the results thus obtained the position of the load that causes no twisting of the channel is easily found, and the location of the shear center for each channel section is thereby determined.

Also, for each of several lateral positions of the load the strains along each of the four edges of the channels were measured at several sections along the beam, the strains being measured over a 2-in. gage line. Thus the effect of the twisting of the channel on the longitudinal stresses at different sections along the beam was determined.

4. *Acknowledgment.*—The main body of the bulletin is the work

*Bauingenieur, Vol. 6, 1925, p. 455.
†Zeitschrift für Math. und Physik., Vol. 58, 1910.

chiefly of the first two authors, and the analysis and results given in the Appendix are due to the last-named author.

The tests herein reported were made as a part of the work of the Engineering Experiment Station, of which Dean M. S. Ketchum is the director, and of the Department of Theoretical and Applied Mechanics, of which Prof. M. L. Enger is the head.

II. Shear Center for Channel Sections

5. *Approximate Location of Shear Center.*—As stated in Section 1 the shear center for a channel section is the point on the axis of symmetry through which the plane of the loads must pass, parallel to the web, to cause the channel beam to bend without twisting, in accordance with the simple flexure theory of beams.

A simple method* of determining the approximate position of the shear center of a channel section is as follows: Let Fig. 1a represent a channel beam (assumed to be a cantilever beam for convenience) subjected to loads P_1, P_2, etc., that lie in a plane, parallel to the web, so located that the loads cause the channel to bend without twisting, and cause stresses on any transverse section in accordance with the flexure formula as indicated by the stresses shown acting on section AB (Fig. 1b).

At any transverse section such as CD the bending stresses and shearing stresses together hold in equilibrium the external forces P_1, P_2, etc., that lie to one side of the section. The vertical shear or resultant of the loads P_1, P_2, etc., is denoted by V. By introducing two equal and opposite forces at the shear center O, each equal to V, the vertical shear may be resolved into a force $V_1 = V$ at O and a couple, the forces of which are V and V_2, constituting a bending couple whose moment is M. This bending moment M is held in equilibrium by the bending stresses on the section CD similar to those shown on section AB (these bending stresses on CD are not shown in Fig. 1b). The vertical force $V_1 = V$ causes an equal (but opposite) resisting shear V_r, which will be assumed to be developed only in the web of the section, and to act along the center line of the web, since the web is narrow. The forces V_1 and V_r form a twisting couple whose moment is Ve, where e is the distance from the center line of the web to the shear center O. For equilibrium this twisting moment

*See A. Ostenfeld, "Teknisk Elasticitetslaere," Fourth Edition (1924) for a similar method.

must be equal (and opposite) to the twisting moment of the lateral shearing forces H in the flanges of the channel. That is,

$$Ve = Hh$$

where h is the distance between the center lines of the flanges, but may be considered without serious error to be the full depth of the section.

A value of H may be found as follows: The difference $T_2 - T_1 = dT$ of the total bending stress in two sections of a flange the distance dx apart causes a longitudinal shearing stress on the area connecting the flange with the web; this area is tdx, where t is the thickness of the flange (in a rolled steel channel t will be taken as the mean thickness of flange). If s_s denotes the shearing unit stress on this area, then, for equilibrium,

$$s_s\, t\, dx = dT$$

But the total bending stress T in one flange at any section is $s\, b_1 t$ where b_1 is the width of the outstanding flange, and $s = \dfrac{M\,\frac{h}{2}}{I}$ is the bending unit stress in the flange; I is the moment of inertia of the whole cross-section of the channel with respect to the centroidal axis parallel to the flanges. Thus

$$T = \frac{M\,h}{2\,I}\, b_1 t$$

and

$$dT = \frac{dM\,h\,b_1 t}{2\,I}$$

where dM is the difference in the bending moments at the two sections. Therefore

$$s_s\, t\, dx = \frac{dM\,h\,b_1 t}{2\,I}$$

or

$$s_s = \frac{dM}{dx} \cdot \frac{h\,b_1}{2\,I} = \frac{Vh\,b_1}{2\,I}$$

in which the familiar relation $\frac{dM}{dx} = V$ is used. As noted, s_s is the value of the longitudinal shearing unit stress on the area connecting the web and flange. But an equal shearing unit stress must exist at the same point on the transverse section of the flange, and this lateral shearing unit stress decreases to zero at the outer edge of the flange. It will be assumed that this variation is a linear one, and hence the average shearing unit stress in each flange is $\frac{V\,h\,b_1}{4\,I}$. The total lateral shear, H, in each flange then is

$$H = \frac{Vhb_1}{4I}\, b_1 t$$

and the moment of these H forces is

$$Hh = \frac{Vb_1^2h^2t}{4\,I}$$

Since the moment Ve must balance this moment in order to prevent the beam twisting we have

$$Ve = \frac{Vb_1^2h^2\,t}{4\,I} \tag{1}$$

But $$I = 2\,b_1t\left(\frac{h}{2}\right)^2 + \frac{1}{12}\,wh^3 = \frac{1}{2}\,b_1th^2\left(1 + \frac{1}{6}\,\frac{wh}{b_1t}\right)$$

in which w is the thickness of the web.

Hence, $$e = \frac{\frac{1}{2}b_1}{1 + \frac{1}{6}\,\frac{wh}{b_1t}} = \frac{\frac{1}{2}b_1}{1 + \frac{1}{6}\,\frac{a_w}{a_f}} \tag{2}$$

in which a_w is the area of the web and a_f is the area of one outstanding flange. This expression shows that e is large when the ratio of the area of the web to the area of the outstanding flange is relatively small, but that the value of e is always less than one-half the width of the outstanding flange. In the tests herein reported e was found to

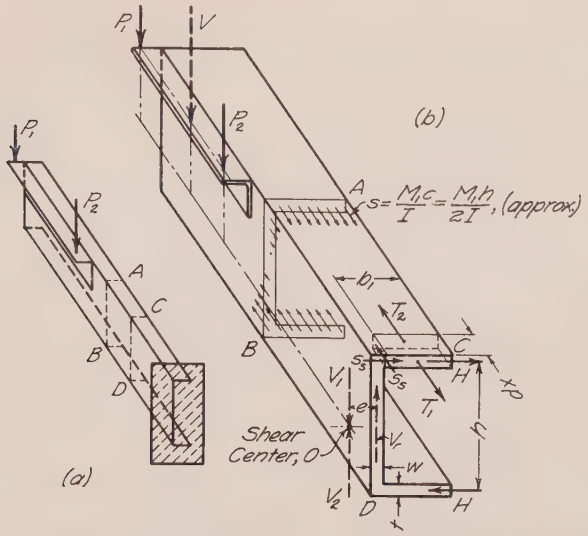


FIG. 1. STRESS IN CHANNEL LOADED THROUGH SHEAR CENTER

vary from approximately 0.20 to 0.36 of the width of the outstanding flange, or from 0.18 to 0.32 of the full width of the flange; and the ratio $\frac{a_w}{a_f}$ varied approximately from 1.7 to 5.7 in the channels tested. The foregoing approximate analysis, although useful, does not yield values of e that agree closely with the experimental value of e for the larger channels; this is discussed further in Section 6.

A slightly more general expression for e is obtained in Section 7 of the Appendix by a different method of analysis. The expression is

$$e = \frac{I'_{xy}\, h}{I_x} \tag{3}$$

in which I'_{xy} is the product of inertia of one of the halves of the cross-section of the channel-shaped beam with respect to rectangular axes passing through the center of the web, one of the axes being an axis of symmetry, h is the distance between the intersections of the center lines of the flanges with the center line of the web, and I_x has the same meaning as has I in Equation (1). Equation (3) can be applied to a section in which the flanges are not perpendicular to the web; it reduces to Equation (2) for the section considered in Fig. 1.

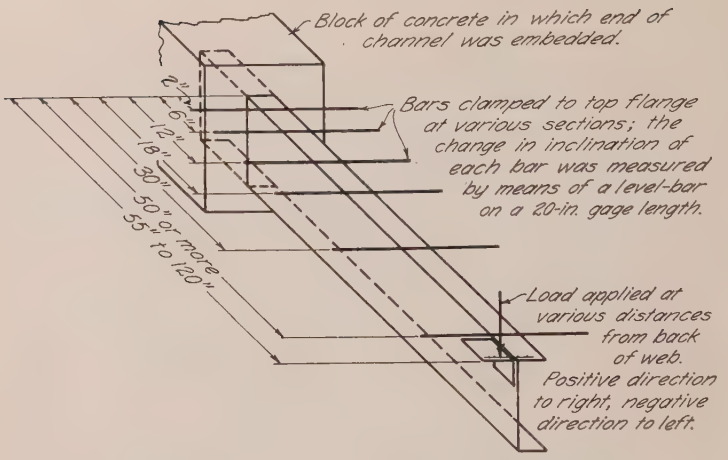

Block of concrete in which end of channel was embedded.
Bars clamped to top flange at various sections; the change in inclination of each bar was measured by means of a level-bar on a 20-in. gage length.
2"
6"
12"
18"
30"
50"
55" to 120"
Load applied at various distances from back of web. Positive direction to right, negative direction to left.

FIG. 2. SKETCH SHOWING ARRANGEMENT OF BEAM FOR TESTING

A still more general method of locating the shear center of a section composed of narrow parts is given in Section 11 of the Appendix. When applied to the channel section it leads to the following expression, which gives values that agree closely with the experimental values,

$$e = \frac{h}{2}\frac{x_c}{y_c} \tag{4}$$

in which x_c and y_c are the coördinates of the centroid of one-half of the channel section with respect to axes having their origin at the center of the web, the x axis being the axis of symmetry; h has the same meaning as in Equation (3).

Values of e for the channels tested are given in Table 1 and are discussed in the next section.

It will be noted that the shear center for the channel is a property of the cross-section; that is, its location depends only on the dimensions of the cross-section of the channel. For other sections composed, like channel sections, of narrow (approximately rectangular) parts but possessing points of symmetry, such as I and Z sections, the point of symmetry of the section is the shear center for the section. Thus the centroid of a section composed of narrow parts is also the shear center for the section, provided that the section has a point of symmetry.

6. *Experimental Determination of Shear Center.*—In order to determine the lateral position of a transverse bending load on a chan-

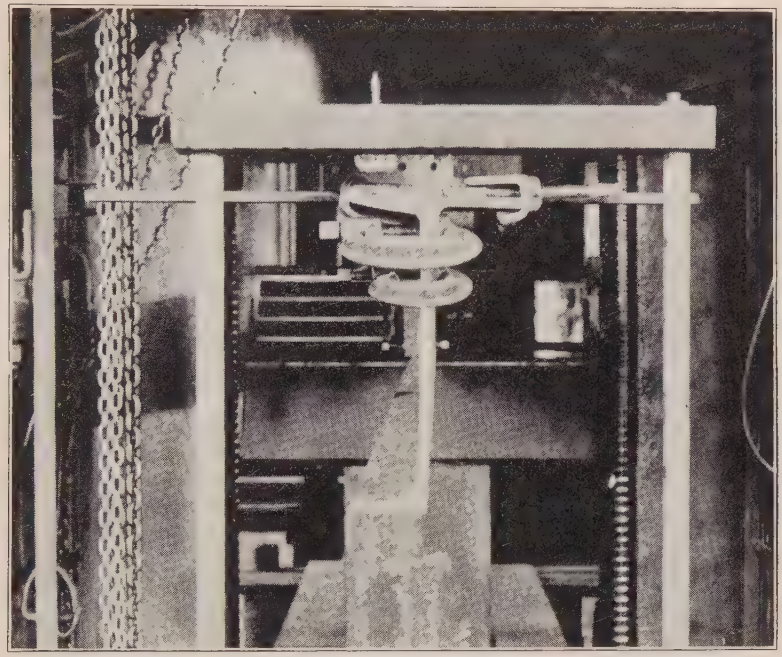


FIG. 3. VIEW OF CHANNEL WITH LOAD APPLIED THROUGH SHEAR CENTER AT END OF CANTILEVER

nel for which no twisting of the channel occurs as it bends, each channel was tested as a horizontal cantilever beam with a vertical load consisting of a suspended weight applied in turn at several lateral positions across the free end of the cantilever. Some of the smaller channels were embedded at one end in a large cube of concrete (see Fig. 2), and this cube was held on the weighing table of a testing machine by applying a load to the top face of the cube by means of the movable head of the testing machine; the section flush with the face of the concrete was restrained by the concrete from warping and remained practically a plane section. The larger channels were tested as double cantilevers (see Figs. 3 and 4); the beam was balanced on a rounded knife edge on the weighing table of a testing machine and equal weights were applied to the two free ends. The central section of the channel was thus maintained a plane section free from warping, and it was further held fixed in position by a light load applied through the movable head of the testing machine to the top flange, and by small horizontal jacks on each side of the top and bottom flanges. In addition, the outstanding flanges at the central section

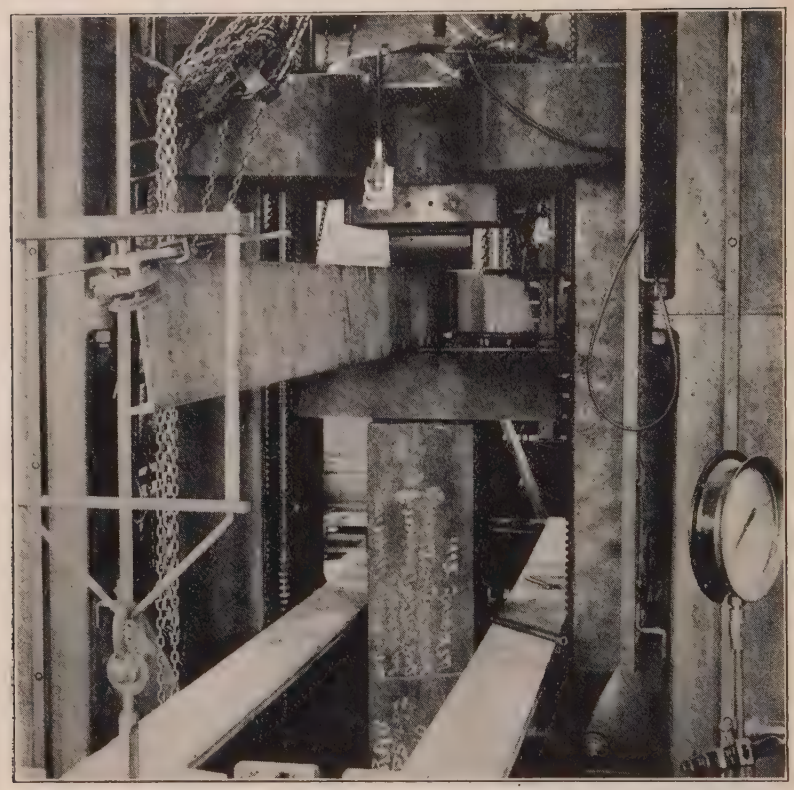


FIG. 4. VIEW OF CHANNEL WITH LOAD APPLIED APPROXIMATELY THROUGH THE CENTROID OF END SECTION

were supported by a vertical stiffener extending between the flanges. It should be noted that this double cantilever is equivalent to a simple beam with a concentrated load applied to the center of the span.

Figure 3 shows a view of a 15-in. 33.9-lb. channel loaded through the shear center, the load being such as to produce a maximum bending stress of 16 000 lb. per sq. in., as computed by the simple flexure formula. It will be noted that the channel bends without twisting.

Figure 4 shows a view of the same channel as is shown in Fig. 3 subjected to a load of the same magnitude applied approximately through the centroid (center of gravity) of the section. The twisting of the beam that accompanies the bending is very evident.

The angle of twist was measured at six or seven sections along the length of the beam for each of several lateral positions of the load at

the end (see Fig. 2). The load was applied at two or three points on each side of the back of the channel; an angle was clamped to the back of the channel at the free end of the beam so that the load could be applied behind the back of the channel.

The angle of twist at each section was measured by means of a level-bar on a 20-in. gage length. For this purpose bars slightly more than 20 in. long were clamped to the top flange of the channel at several sections along the length of the beam (the bars near the free end are shown in Figs. 3 and 4), and the changes in inclination of these bars were measured on a 20-in. gage length; the change in elevation of one end of the gage length with respect to the other was measured to 0.001 in. by adjusting a micrometer screw at one end of the level-bar. The bubble in the level-bar was a 20-second bubble, which was amply sensitive.

In Fig. 5 are shown the experimental results indicating the relation between the angle of twist and the distance along the beam.

For example, in the case of the 4-in. 5.4-lb. channel (Fig. 5a) each section of the channel (except the fixed section) twisted clockwise when the load was applied one inch in front of (to the right of) the back of the web (+1.0 in.), the twist increasing with the distance from the fixed end. Likewise the channel twisted clockwise when the load was applied one-half inch in front of the back of the web (+0.5 in.), and also when applied at the back of the web. But when the load was applied at one-half inch behind the back of the web (−0.5 in.) the channel twisted slightly counter clockwise, and hence it is clear that if the load were placed slightly less than one-half inch behind the back of the web the channel would not twist as it bends.

The definite location of the shear center for each channel, however, was determined by plotting curves showing the angles of twist at each of several sections along the beam, and corresponding lateral positions of the load. These curves are shown in Fig. 6.

For example, Fig. 6a shows that in the case of the 4-in. 5.4-lb. channel beam no twisting of the beam occurred at any of the four sections indicated, at various distances from the fixed end of the cantilever, when the load at the free end of the cantilever was applied approximately 0.40 in. to the rear of the back of the web or approximately 0.50 in. from the center line of the web.

It will be noted from Fig. 6 that the location of the shear center for several of the channel sections seems to vary somewhat, depending on the position of the section at which the angles of twist are measured, the twisting of the specimen closest to the fixed end (6 in. from

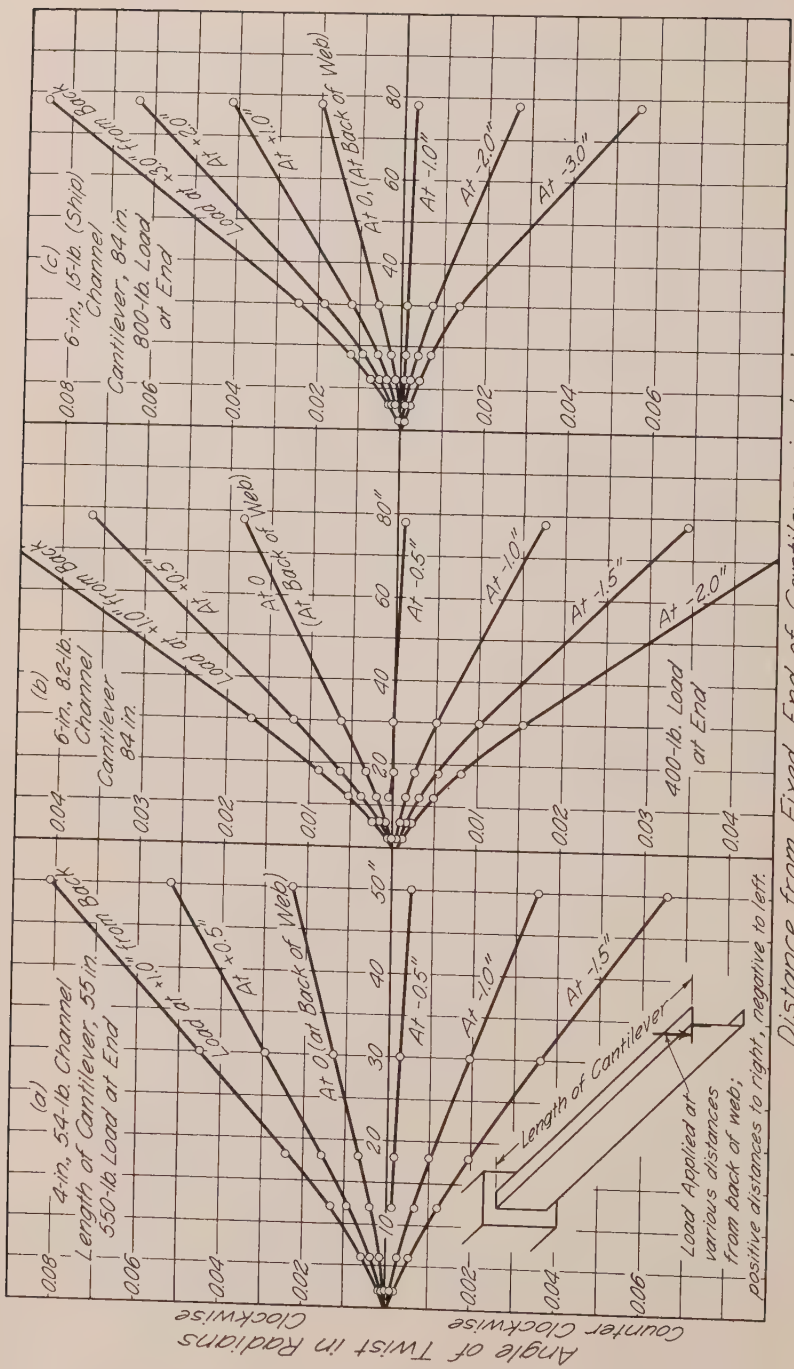


FIG. 5. CURVES SHOWING RELATION BETWEEN ANGLE OF TWIST AND DISTANCE FROM FIXED END OF CANTILEVER

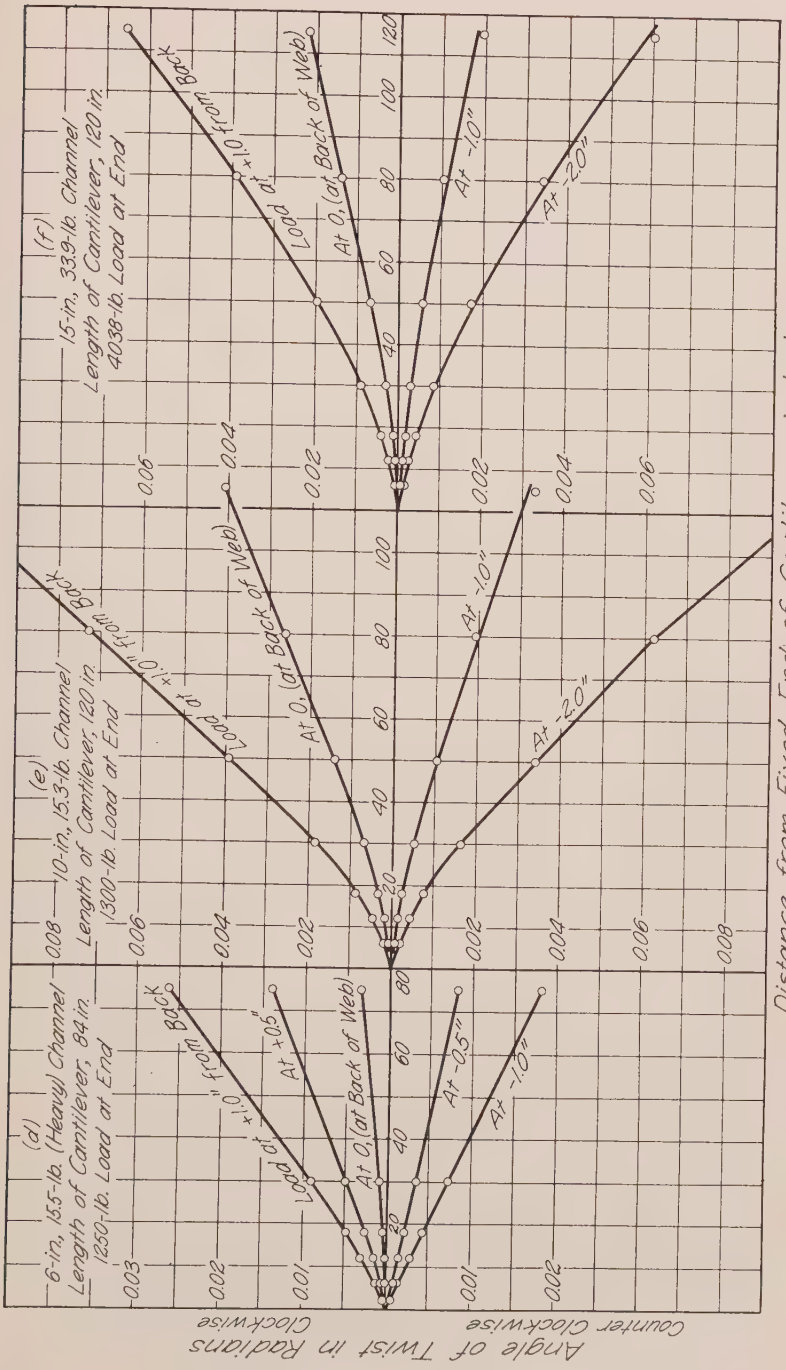


FIG. 5 (CONCLUDED). CURVES SHOWING RELATION BETWEEN ANGLE OF TWIST AND DISTANCE FROM FIXED END OF CANTILEVER

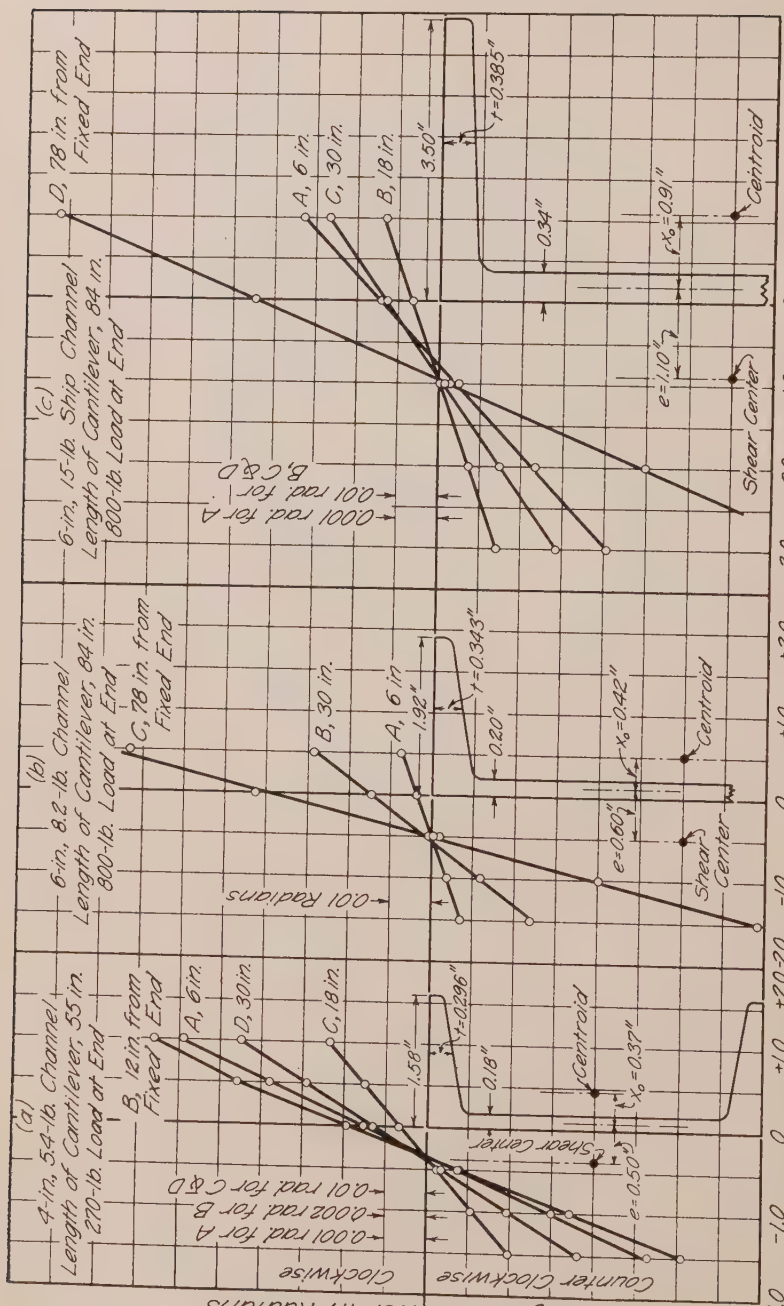


FIG. 6. LOCATION OF SHEAR CENTER DETERMINED FROM TEST DATA

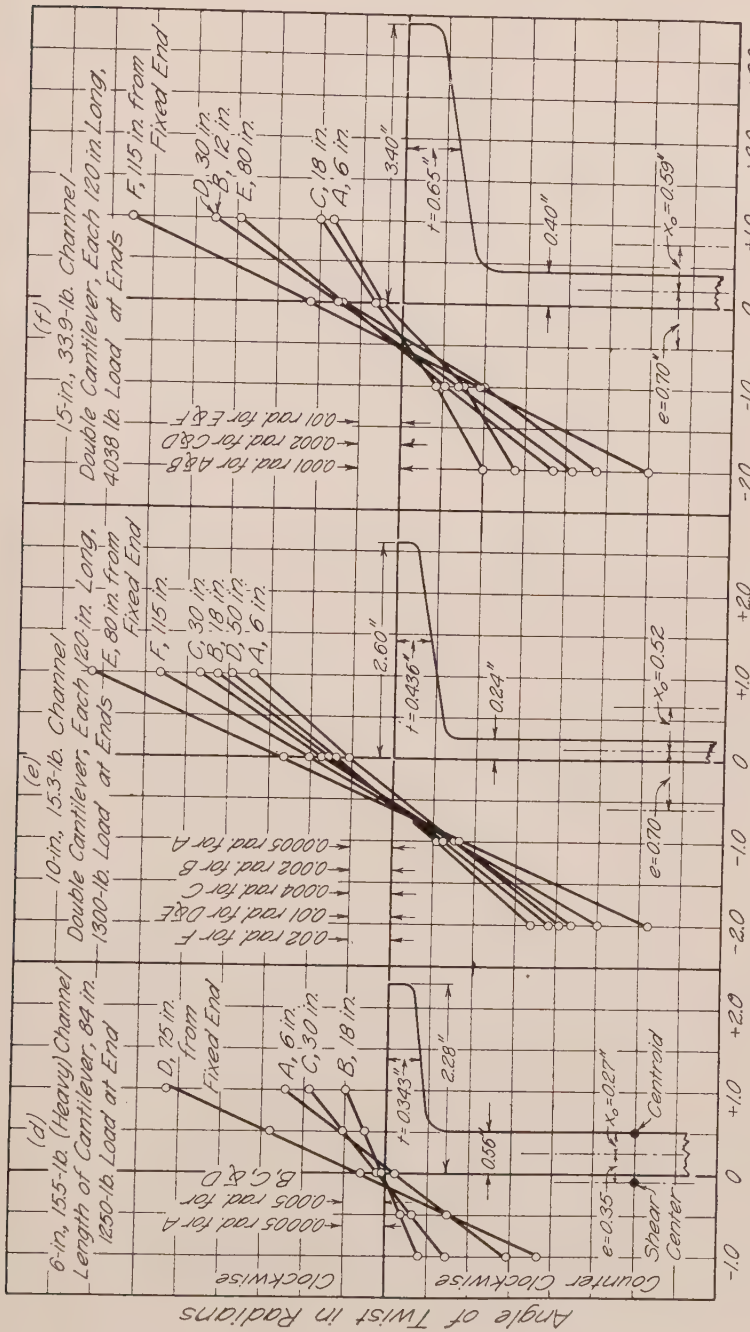
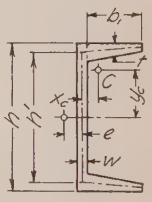


FIG. 6 (CONCLUDED). LOCATION OF SHEAR CENTER DETERMINED FROM TEST DATA

Table 1

Experimental and Calculated Values of the Distance of Shear Center from Center Line of Web

(All dimensions in inch units)

Size of Channel	Ratio $\frac{wh}{b_1 t}$		Experimental Values of e	Values of e in $e = \frac{\frac{1}{2}b_1}{1 + \frac{1}{6}\frac{wh}{b_1 t}}$		Values of e in $e = \frac{h'}{2}\frac{x_c}{y_c}$		
	t.d.*	m.d.†		t.d.*	m.d.†	t.d.*	m.d.†	
4 in.–5.4 lb.	1.73	1.68	0.50	0.54	0.56	0.46	0.47	
6 in.–8.2 lb.	2.04	2.04	0.60	0.64	0.64	0.54	0.56	
6 in.–15.3 lb. (Ship)	1.67	1.85	1.10	1.23	1.14	1.14	1.05	
6 in.–15.5 lb. (Heavy)	5.70	5.25	0.35	0.44	0.45	0.40	0.42	
10 in.–15.3 lb.	2.10	2.44	0.70	0.86	0.87	0.69	0.70	
15 in.–33.9 lb.	3.08	3.33	0.70	1.00	0.95	0.81	0.78	

*From hand book tabular dimensions.
†From measured dimensions.

the fixed end) indicating a location of shear center closer to the back of the web than is shown by the twist of sections more remote from the fixed end.

In this connection the following conditions should be considered: The angles of twist of a section close to the fixed end are relatively small and hence the measured values of these angles are more likely to be in error than those of sections more remote from the fixed end. Further, the areas of the two flanges of the 6-in. 15.5-lb. (heavy) section differed by as much as 10 per cent; also, the larger (10-in. and 15-in.) sections were noticeably crooked as received, and, although they were straightened considerably before testing, a slight crookedness remained.

In Table 1 are given values of the distances of the shear centers of the various channel sections from the center lines of the webs of the channels as obtained from Fig. 6. In the same table are also shown the values of these distances as obtained from the formulas for e given in Section 5, using handbook tabular values for the dimensions of the sections.

The values in Table 1 indicate that the simple analysis given in Section 5 leading to Equation (2) for e, although a convenient and useful method for obtaining approximate values, may yield results considerably in error for the larger channels due, probably, to the simplifying assumptions made in the derivation. Table 1 indicates

that Equation (4) gives values of e in close agreement with the values found from the tests.

7. *Observed Values of Longitudinal Stress in Channel When Loaded through Shear Center.*—If the load on the channel is applied through the shear center, the channel bends without twisting, as is assumed in the simple flexure theory, and hence the longitudinal stresses at any section are due only to the vertical bending moment at the given section, and are given by the simple flexure formula, $s = \frac{Mc}{I}$, the axis of symmetry being the neutral axis.

In order to determine by tests the longitudinal stresses at any section of the channel beam, the strains were measured along each of the four edges of the channel at each of six or more sections along the beam, the sections selected being in general those at which the angles of twist were measured (see Fig. 2). The values of the longitudinal stresses were computed from the strain measurements by using a value of 30 000 000 lb. per sq. in. for the modulus of elasticity. Readings of the strain gage could be repeated with a maximum error of ± 1000 lb. per sq. in.

Curves showing the relation between the longitudinal stress along each of the four edges of the channel and the distance from the fixed section of the beam are given in Fig. 7, the load being applied through the shear centers as found from Fig. 6. The broken lines represent the values of s obtained from the flexure formula $s = \frac{Mc}{I}$, the load applied being such as to produce a value of s in the outer fibers at the fixed end of the beam equal to 16 000 lb. per sq. in. The gage lines for the strain measurements, however, were not located at the extreme outer fibers but on the back of the web and on the edge of the flange about one-eighth of an inch from the outer fibers.

The curves in Fig. 7 show that all the channels (except the 6-in. 15.5-lb. heavy section), when loaded through their shear centers, developed longitudinal stresses that are given satisfactorily by the simple flexure formula.

In the case of the 6-in. 15.5-lb. heavy channel the difference in stress along the edges A and C (Fig. 7c) may be due in part to the difference in areas of the flanges of this section; this would cause a condition of unsymmetrical bending in which the neutral axis is inclined to the horizontal axis, thereby causing A to be closer to and C farther from the neutral axis than is assumed.

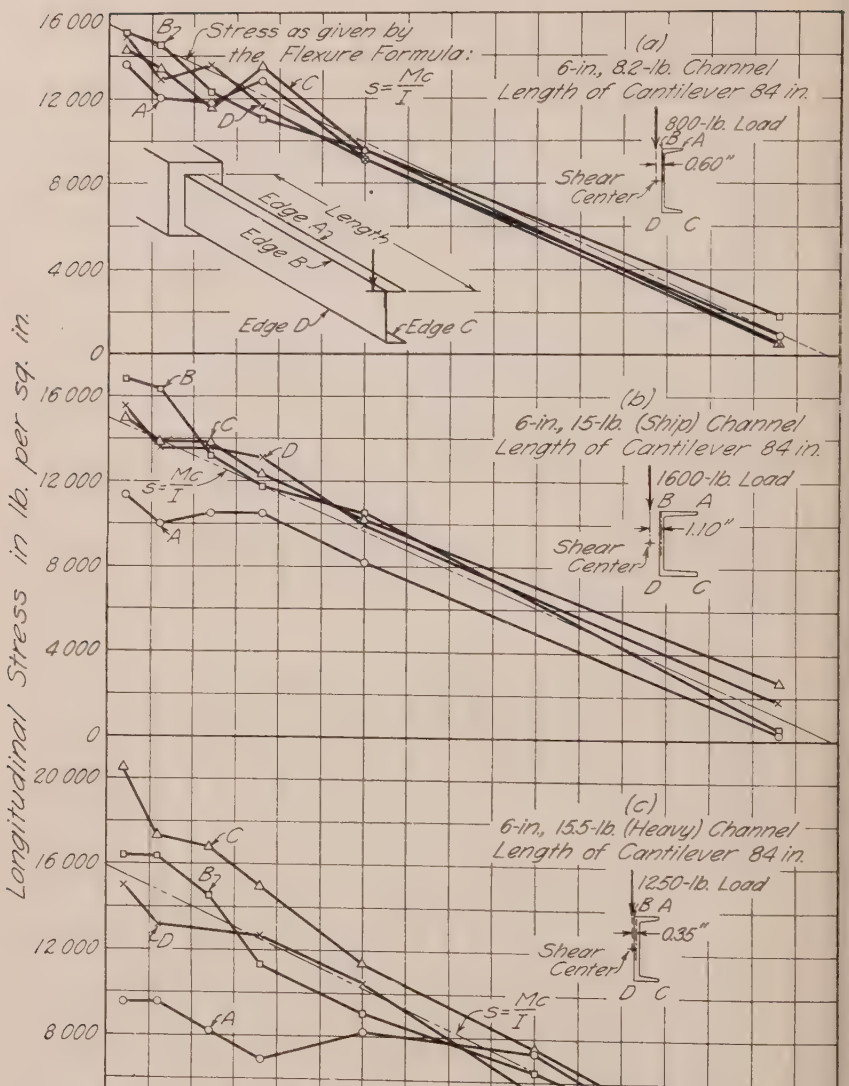

Longitudinal Stress in lb. per sq. in.
Distance from Fixed End of Cantilever in Inches
Stress as given by the Flexure Formula: s = Mc/I
(a) 6-in., 8.2-lb. Channel Length of Cantilever 84 in.
800-lb. Load
0.60"
Shear Center
Edge A
Edge B
Edge D
Edge C
Length
(b) 6-in., 15-lb. (Ship) Channel Length of Cantilever 84 in.
1600-lb. Load
1.10"
(c) 6-in., 15.5-lb. (Heavy) Channel Length of Cantilever 84 in.
1250-lb. Load
0.35"

FIG. 7. OBSERVED LONGITUDINAL STRESS IN CHANNEL LOADED THROUGH SHEAR CENTER

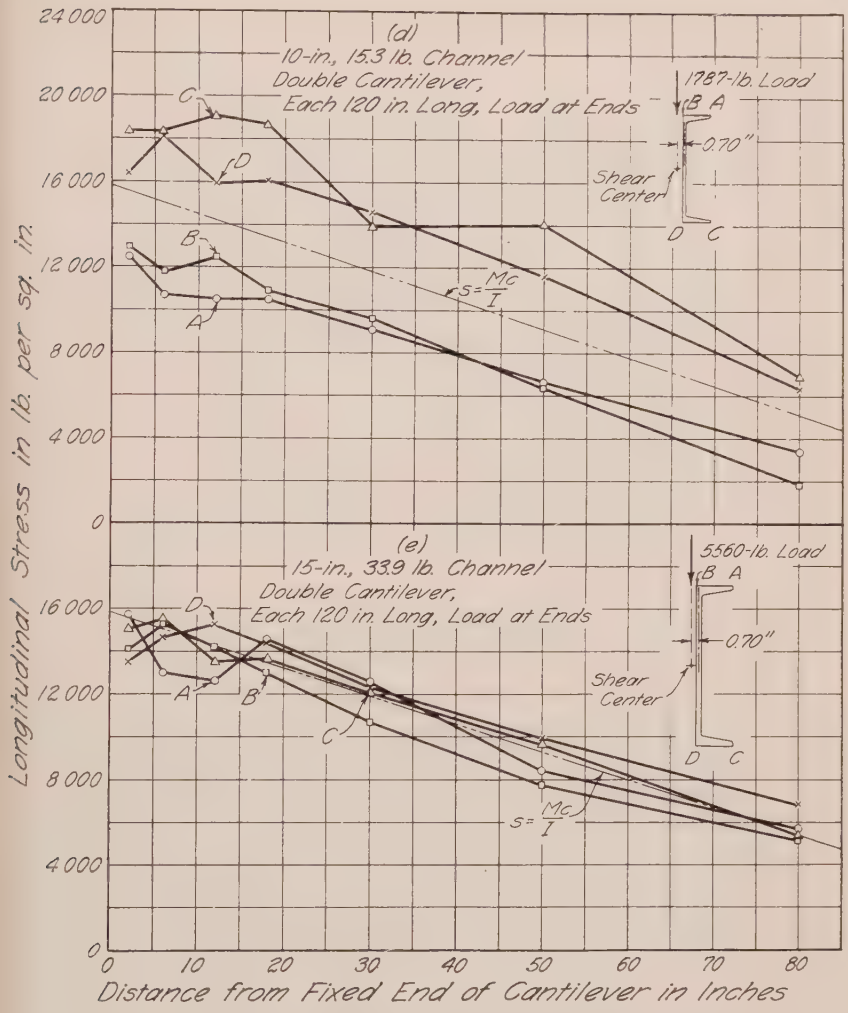


FIG. 7 (CONCLUDED). OBSERVED LONGITUDINAL STRESS IN CHANNEL LOADED THROUGH SHEAR CENTER

III. LONGITUDINAL STRESS IN CHANNEL WHEN THE TRANSVERSE LOADS PRODUCE TWISTING WITH THE BENDING

8. *Influence of Twisting on Longitudinal Stresses.*—Let a channel (Fig. 8) be loaded through some point other than the shear center. This load P may be resolved into an equal load through the shear center and a twisting couple whose moment is Pm. Now the load through the shear center produces bending without twisting, as dis-

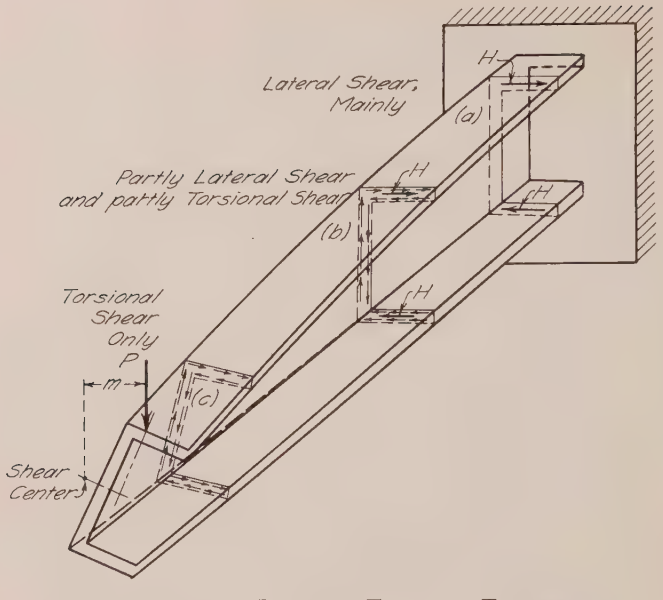


FIG. 8. GENERAL TORSIONAL EFFECT OF TRANVERSE LOAD ON CHANNEL BEAM

cussed in the preceding sections, and the bending moment due to this load is held in equilibrium at any section by a resisting moment due to the longitudinal stresses as given by the simple flexure formula. The external twisting moment Pm develops also on the section a resisting twisting moment due to shearing stresses that holds in equilibrium the external twisting moment, but the shearing stresses producing this resisting moment develop quite differently on sections near the restrained end and on sections near the free end of the beam.

The sections near the free end of the beam can twist because they are free to warp, for, as shown by Saint Venant in his classical memoir on the Torsion of Prismatic Bars, the twisting of all bars except right cylindrical bars must be accompanied by a warping of the sections when the bar is subjected to a twisting moment, and by so doing there is developed a resisting torsional moment due to shearing stresses, distributed in the case of a channel section somewhat as shown at (*c*) Fig. 8. When a channel section is free to warp, then, the twisting moment does not affect appreciably the longitudinal stresses in the beam at this section but merely produces shearing stresses on the section, except for very thin sections and large angles of twist.

If, however, a section is restrained from warping, as is the case at the central section of a simple beam symmetrically loaded, or at the fixed end of a cantilever beam, then torsional shearing stresses like those shown in Fig. 8c cannot develop. The resisting moment at the section is developed by the lateral shearing stresses H in the flanges (as at (a), Fig. 8) which are accompanied by lateral bending of the flanges, that is, bending of the flanges (in opposite directions) in their own planes. This lateral bending of each flange causes a longitudinal tensile stress along one edge and a compressive stress along the other edge of each flange which must be added (algebraically) to the longitudinal stress caused by the vertical bending due to the load acting at the shear center in order to obtain the total longitudinal stress on the edge.

The maximum additional longitudinal stress in some of the channels tested, which occurs at the section that is restrained from warping, and hence from twisting, was more than one-half of the longitudinal stress caused by the vertical bending, when the load was applied approximately through the centroid of the end section of the cantilever.

The curves in Fig. 5 giving the relation between the angle of twist and the distance from the fixed end of the beam show clearly that the effect of restraining a section from warping and hence from twisting extends a short distance out from the restrained section, diminishing more and more rapidly with the distance until the effect is practically nil; the remainder of the beam twists approximately a constant amount per unit length. At any section at some distance from the restrained section (as at (b), Fig. 8) the resisting moment that holds the external twisting moment in equilibrium is made up partly of a torsional shearing moment developed by the twisting (and warping) of the section, and partly by the moment of the lateral (flexural) shear in the flanges that accompanies the lateral bending of the flanges.

Figure 8, then, illustrates approximately the difference in elastic action and the dominant shearing stresses in different portions of a channel beam which is bent by a transverse load not acting through the shear center. Lateral shear only (or mainly) develops at a section very near to the fixed section (as illustrated at (a), Fig. 8) and is accompanied by additional longitudinal stress along one edge of each flange. At (b) of Fig. 8 there is some warping of the section and hence some torsional shearing stress developed, and some lateral or bending shear; at (c) there is practically no effect of the restraint of the fixed

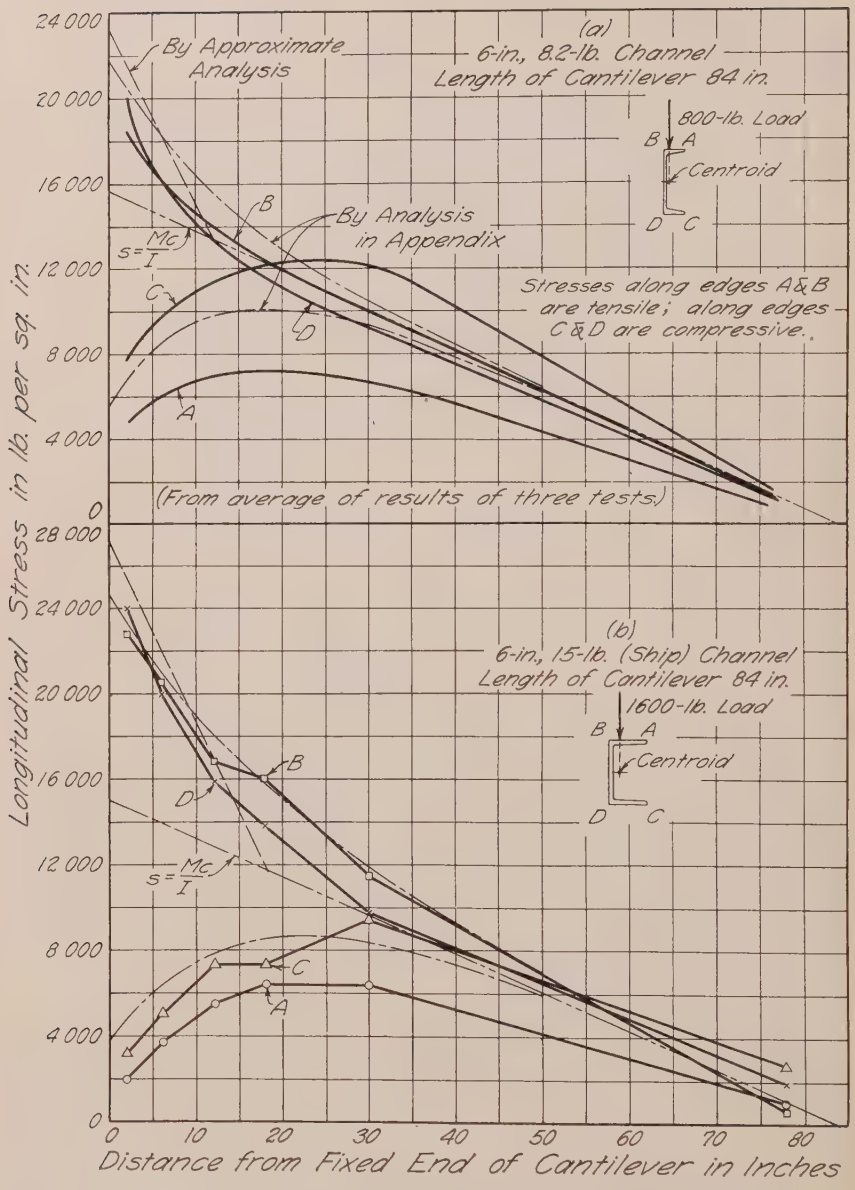


FIG. 9. OBSERVED LONGITUDINAL STRESS IN CHANNEL LOADED APPROXIMATELY THROUGH CENTROID

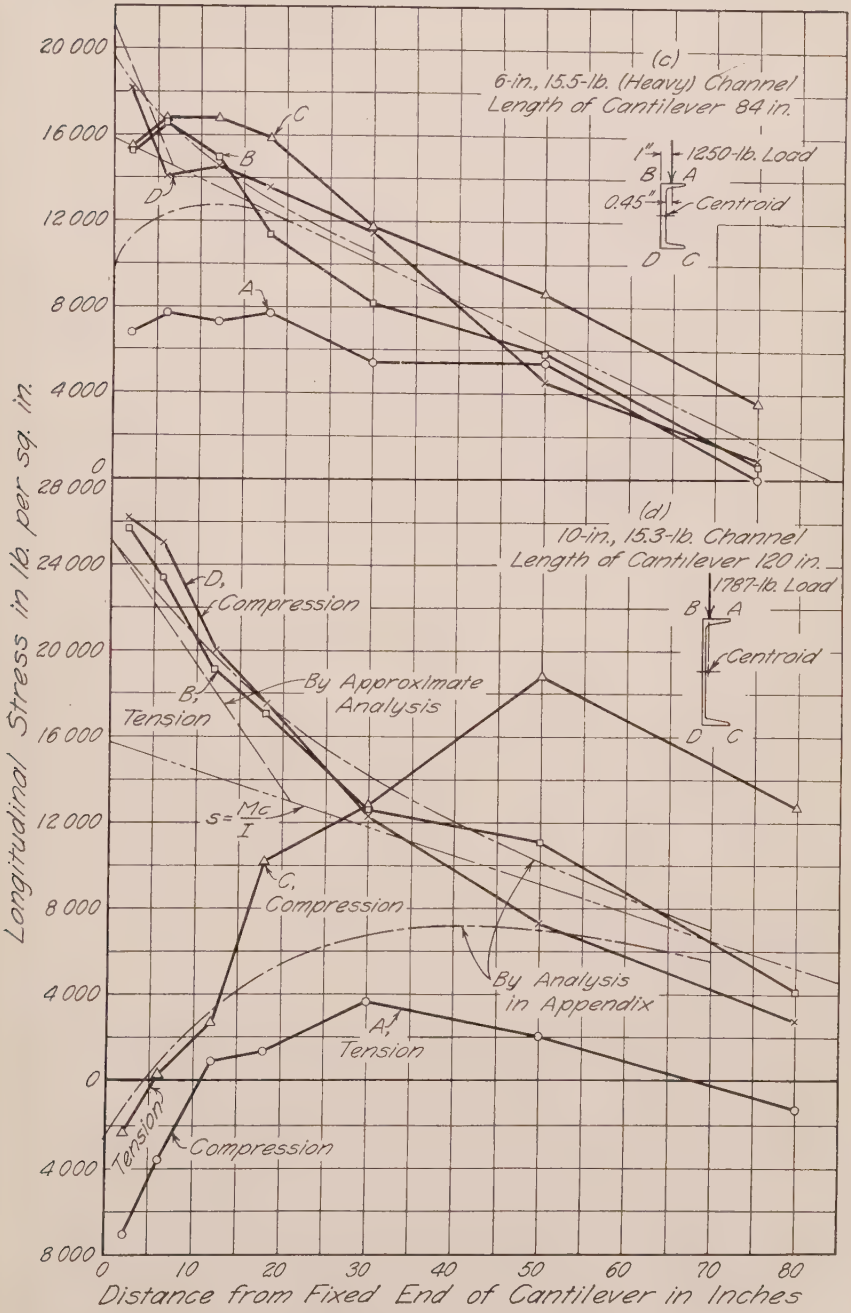


FIG. 9 (CONTINUED). OBSERVED LONGITUDINAL STRESS IN CHANNEL LOADED APPROXIMATELY THROUGH CENTROID

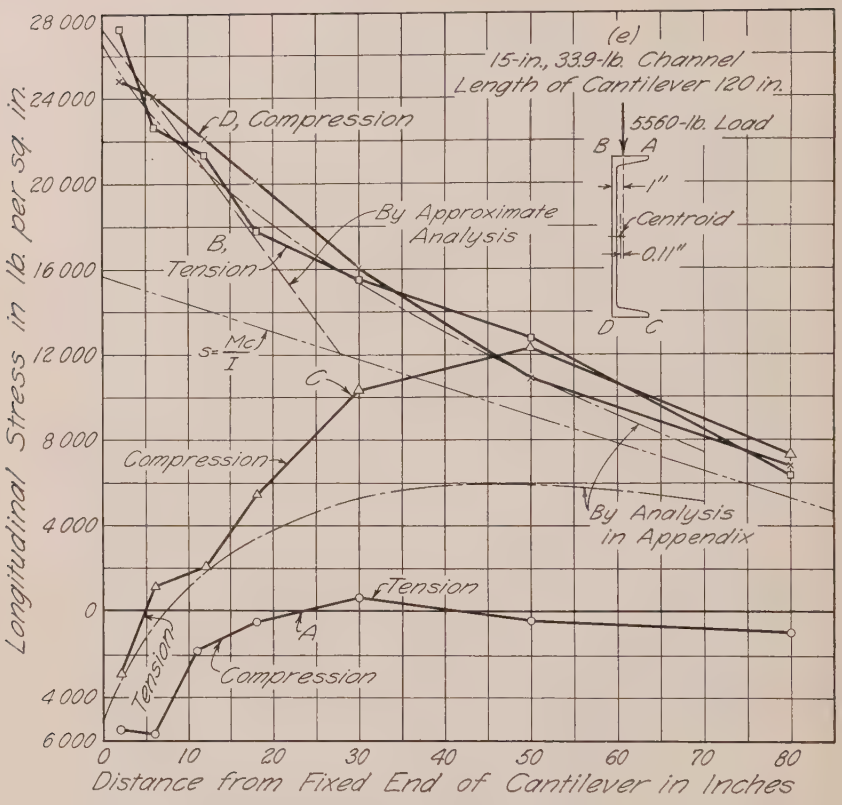


FIG. 9 (CONCLUDED). OBSERVED LONGITUDINAL STRESS IN CHANNEL LOADED APPROXIMATELY THROUGH CENTROID

section and hence only torsional shear is developed, with no increase in longitudinal stress due to the twisting.

9. *Observed Values of Longitudinal Stress in Channels When Load Did Not Pass through Shear Center.*—The values of the stresses found from the strain-gage readings at various sections of each channel tested, when the load did not pass through the shear center of the end section, are given by the curves in Figs. 9 and 10. Strains were measured in all the channels except the 4-in. 5.4-lb. channel. The strain measurements were made on a 2-in. gage length at each edge at sections 2 in., 6 in., 12 in., 18 in., 30 in., and 50 in. or more from the fixed end of the cantilever. In general the curves in Fig. 9 give the longitudinal stresses in each channel caused by a load acting in front (to the right) of the shear center, approximately through the centroid

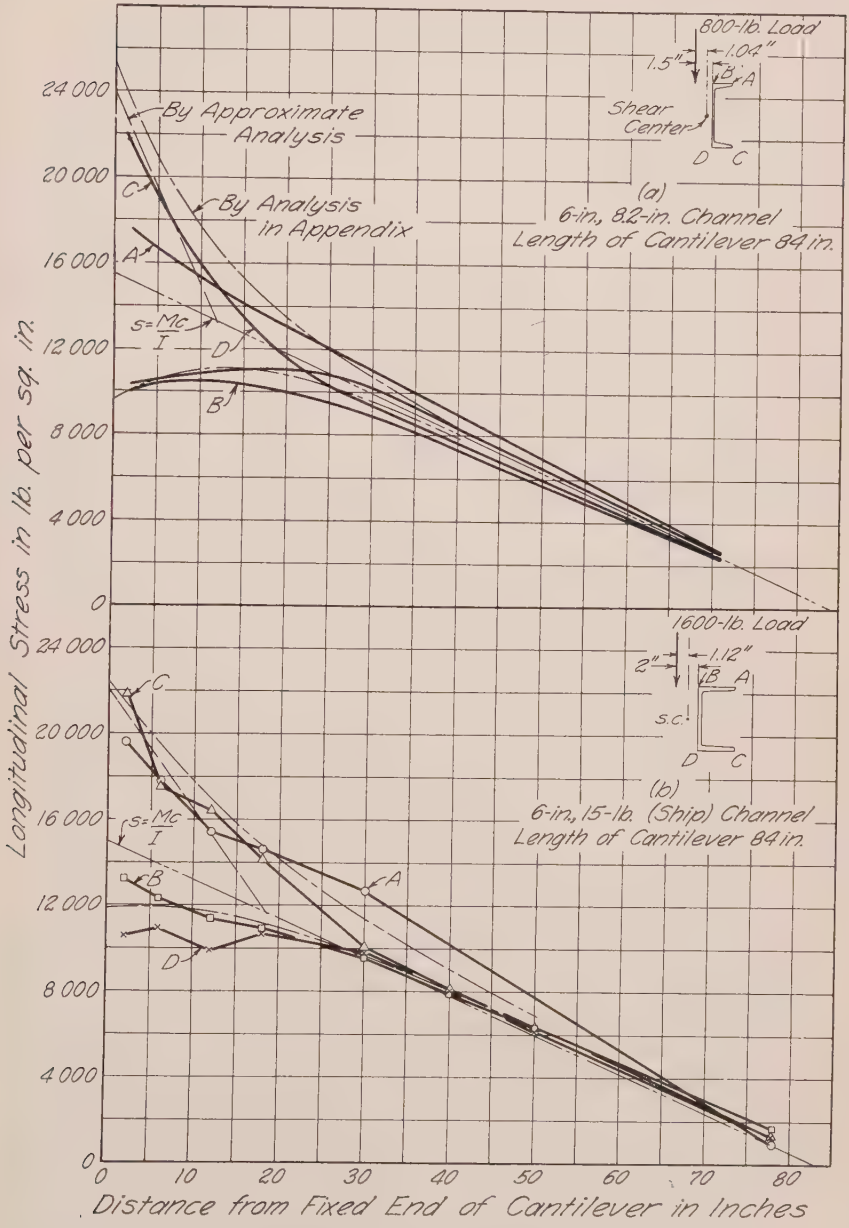


FIG. 10. OBSERVED LONGITUDINAL STRESS IN CHANNEL LOADED BACK OF SHEAR CENTER

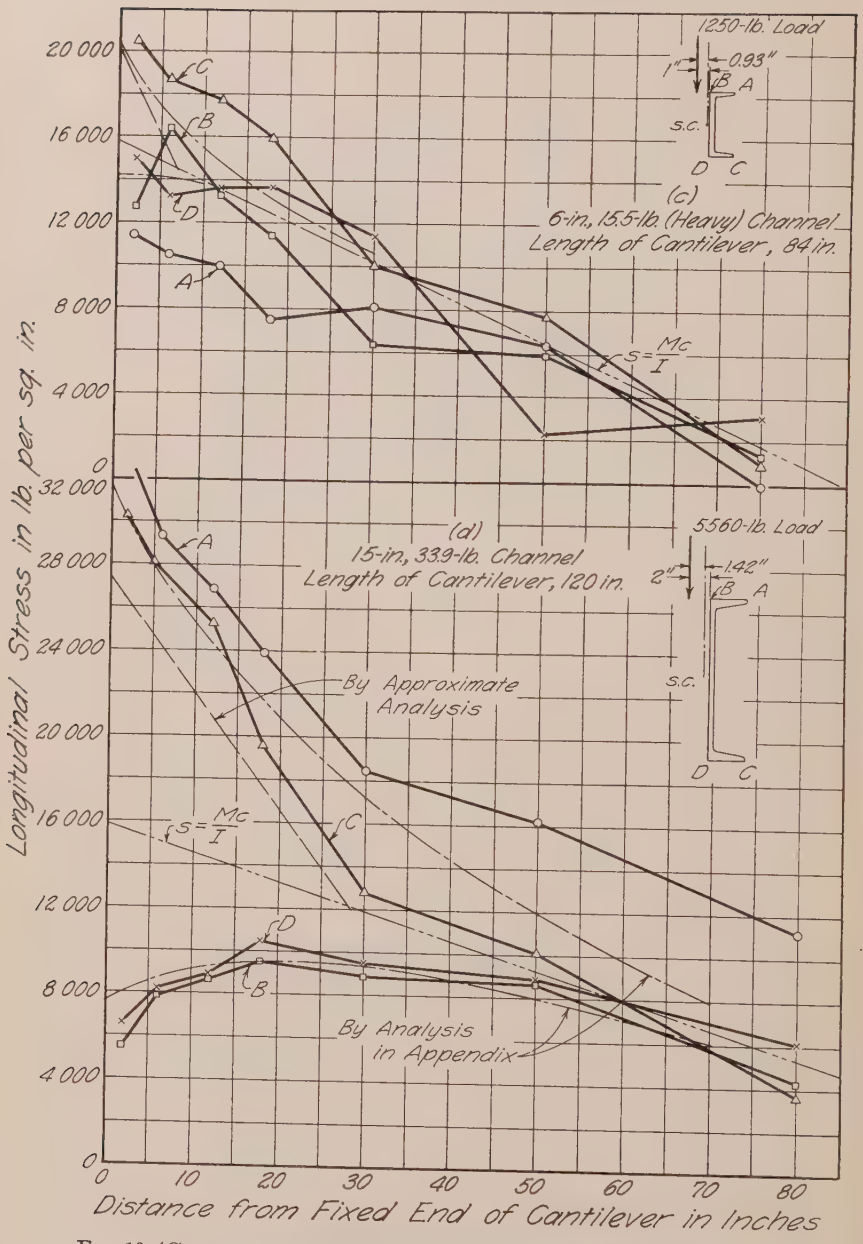


FIG. 10 (CONCLUDED). OBSERVED LONGITUDINAL STRESS IN CHANNEL LOADED BACK OF SHEAR CENTER

of the section, and the curves in Fig. 10 give the longitudinal stresses in each channel caused by a load acting behind (to the left of) the shear center.

In Fig. 9 the action line of the load is shown as passing through the centroid of the channel section (except in Fig. 9c). The load, however, was applied on the upper flange at a point directly above the position of the centroid corresponding to no load. Thus a very small load causing a negligible angle of twist would act through the centroid of the section, but a larger load causing an appreciable angle of twist would not act through the centroid (see Fig. 4). Nevertheless, a load thus applied will be referred to as acting through the centroid or as acting approximately through the centroid.

The load applied to the channel in each case was such as to give, according to the flexure formula, a value of s of 16 000 lb. per sq. in. in the most remote fibers at the fixed end of the beam; the values of s, however, at the gage lines used for measuring strains were somewhat less than at the most remote fiber.

As discussed in Section 7, when the load was applied through the shear center of the channel the observed stresses agreed well with the values of s obtained from the flexure formula. When the load was applied to either side of the shear center, however, twisting accompanied the bending of the channel, which caused additional longitudinal stresses along two edges as explained in Section 8.

For example, the curves in Fig. 9b for the 6-in. 15.3-lb. ship channel loaded approximately through the centroid of the section show that the longitudinal tensile stress along edge B at the fixed section is approximately 25 000 lb. per sq. in., that is, the additional longitudinal stress due to the twisting and accompanying lateral bending of the top flange caused by a load applied through the centroid is about 10 000 lb. per sq. in., since the stress on the same fiber due to a load applied through the shear center which causes no twisting is about 15 000 lb. per sq. in., as given by the flexure formula. Also the stress along edge A at the fixed section is decreased by the lateral bending by about the same amount, or somewhat more.

Similarly, the compressive stress along D at the fixed section is increased and that along C decreased, due to the lateral bending of the bottom flange.

As shown in Fig. 10b, when the load is applied to the left of the shear center the stresses along A and C are increased and those along B and D decreased by the lateral bending of the flanges.

Further, the curves in Figs. 9 and 10 show that the maximum additional longitudinal stress occurs at the fixed section, and that the

additional stress decreases rather rapidly with the distance from the fixed section, becoming negligible at about ¼ to ⅓ of the length of the beams tested. However, the distance from the fixed section within which the lateral bending is measurable depends not on the length of the beam, but on the relation of the lateral-bending stiffness of the flanges to the torsional stiffness of the channel section. And, theoretically, the influence of the lateral bending extends to the free end of the beam.

The theoretical values of the longitudinal stresses in the channels, as found from the analysis given in the Appendix, are shown by the dash and dot lines in Figs. 9 and 10, and the values of the additional longitudinal stress according to the simple approximate analysis given in the following section (Section 10) are shown by the straight broken lines.

In all cases the theoretical stresses along the two edges *A* and *C* are equal, and the same condition holds for edges *B* and *D*. The cause of the rather wide variations in the stresses observed in several of the channels along pairs of edges that theoretically should be stressed equally, is not clear. The rather erratic results for the 6-in. 15.5-lb. heavy channel (Figs. 9c and 10c) are probably due mainly to the variation in flange areas, a condition already referred to in Section 7.

Taken as a whole, the observed values of the longitudinal stresses are in substantial agreement with the theoretical values of these stresses, and the maximum values of the longitudinal stresses are given satisfactorily by the simple approximate analysis discussed in the following section.

10. *Simplified Approximate Analysis of Longitudinal Stresses.*— A rather complete analysis of the longitudinal stresses in a channel beam subjected to transverse loads that do not pass through the shear center of the section is given in the Appendix.

The following simple analysis leads to approximate but useful results: Fig. 11a represents a typical curve (see Fig. 5) showing the relation between the angle of twist of the channel beam and the distance from the fixed section of the beam. In Fig. 11a let a denote the distance from the fixed section to the section AB (Fig. 11b) at which the straight line portion of the curve intersects the horizontal axis. In the length $(l\text{-}a)$ of the beam it will be assumed that the twisting moment Pm causes only pure torsion, and hence that only torsional shearing stresses are developed, as at (c), Fig. 8; in the length a it will be assumed that only lateral shearing stresses are developed

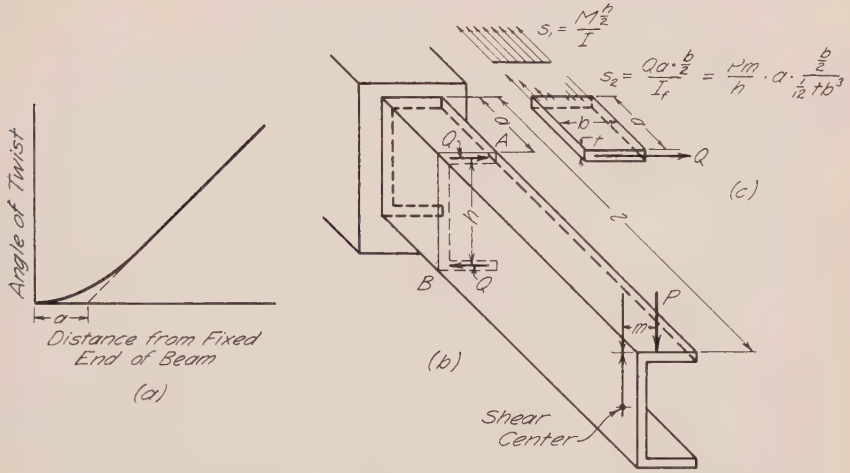


FIG. 11. APPROXIMATE ANALYSIS OF LONGITUDINAL STRESS IN CHANNEL WHEN LOAD IS NOT APPLIED THROUGH SHEAR CENTER

accompanied by the lateral bending of the flanges, as at (*a*) of Fig. 8. The change from torsional shear to lateral shear, of course, is gradual as shown by the curve in Fig. 11a, and as explained in Section 8, but the foregoing simplifying assumption does not lead to a serious error in determining the maximum additional longitudinal stress caused by the twisting of the beam.

Let the lateral shear in each flange at section AB be denoted by Q, then since the moment Qh of these lateral shears must be equal to Pm we have

$$Q = \frac{Pm}{h} \tag{5}$$

where m is the distance from the shear center to the action line of the load P, and h may be considered to be the depth of the channel.

The force Q causes each flange of length a to bend laterally, causing a longitudinal stress s_2 at each edge of the flange, tensile stress at one edge, and compressive stress at the other (Fig. 11c). Assuming that the lateral bending of each flange is in accordance with the flexure formula, and that each flange has a rectangular cross-section, we have

$$s_2 = \frac{M\frac{b}{2}}{I_f} = \frac{Pm}{h}\, a\, \frac{\frac{1}{2}\, b}{\frac{1}{12}\, tb^3} = \frac{6\, P\, m\, a}{htb^2} \tag{6}$$

in which I_f is the moment of inertia of the flange area, t is the mean thickness of the flange, and b is the full width of the flange.

This value of s_2 is an approximate value of the additional longitudinal stress caused by virtue of the fact that the load P is not applied through the shear center but at the distance m therefrom. To this value of s_2 must be added (algebraically) the stress s, due to the vertical bending moment, to obtain the total unit stress at the edge of the flange, as is indicated in Fig. 11c.

The values of the additional longitudinal stress as given by this simplified approximate analysis are represented in Figs. 9 and 10 by the straight broken lines. The value of a for each channel was found as follows:

The value for a given in the Appendix, Equation 35, shows that it depends on the relative values of the lateral-bending stiffness and the torsional stiffness of the section, and that it may be given approximately by the expression

$$a = k\,h\,\sqrt{\frac{EI_y}{E_s J}} \tag{7}$$

in which k is a constant, E and E_s are the tensile and shearing moduli of elasticity, respectively, I_y is the moment of inertia of the channel section with respect to a centroidal axis parallel to the web, and J is an equivalent polar moment of inertia of the channel section. For a section made up of slender approximately rectangular areas, such as a rolled steel channel, angle, or I section, J is given approximately by the expression

$$J = \sum \tfrac{1}{3}\, ld^3 \tag{8}$$

in which l is the long dimension of each rectangular area, and d is the mean thickness, or mean short dimension.

For steel the relation $E = 2.5E_s$ may be used, and the expression for a in Equation (7) if written

$$a = 0.6\,h\,\sqrt{\frac{I_y}{J}} \tag{9}$$

gives values, as shown in Table 2, that are in good agreement with those found by the analysis in the Appendix, and in fair agreement with those found from experiment. It should be noted, however, that the values of a found from the test data are subject to consider-

TABLE 2

VALUES OF a

(All dimensions in inch units)

Size of Channel	I_y (from handbook) (inch^4)	$J = \Sigma \frac{1}{3} l d^3$ (inch^4)	$a = 0.6h\sqrt{\frac{I_y}{J}}$	Value of a from Eq. 35a App.	Value of a from Tests
4 in.–5.4 lb	0.32	0.032	7.6	7.7	7.2
6 in.–8.2 lb	0.70	0.062	12.0	11.5	8.8
6 in.–15.3 lb. (Ship)	5.1	0.198	18.2	16.8	10.3
6 in.–15.5 lb. (Heavy)	1.3	0.398	6.5	7.3	5.5
10 in.–15.3 lb	2.3	0.177	21.0	23.2	24.3
15 in.–33.9 lb	8.2	0.870	27.6	28.8	25.0

able error because a small change in slope of the straight line portion of the curve (Fig. 11a) causes a large change in the value of a, and for several of the beams the number of plotted points for determining the straight line portion of the curve was insufficient. For this reason the data represented in Fig. 5 were plotted on logarithmic paper; the relation between angle of twist and distance along the beam when so plotted should theoretically be represented by a straight line, and hence the values of angles of twist on the curved portion of the curves in Fig. 5 could be used to test the consistency of the few points on the straight line portions of the curves. In this manner more consistent and reliable values of a were obtained than could be obtained directly from the curves plotted from the observed values of angle of twist, similar to those of Fig. 5.

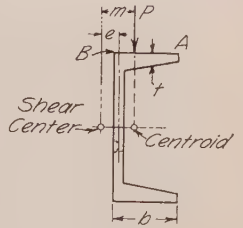
11. *Additional Longitudinal Stresses.*—Table 3 gives values of the maximum additional longitudinal stress in the various channels caused by loads applied to the right of (in front of) the shear centers, approximately through the centroids of the sections. The values of the maximum additional stress in Table 3 were obtained (1) by the simple approximate method (Equation (6)), using values of a from Equation (5); (2) from the observed stresses as estimated from the curves in Fig. 9; and (3) from the analysis developed in the Appendix. Similarly, the maximum additional longitudinal stresses caused by loads applied to the left of (to the back of) the shear centers are given in Table 4.

The agreement in the values of the maximum additional longitudinal stresses obtained by the three methods, although not close in all cases, is considered to be satisfactory. A comparison of the stresses at any distance from the fixed section as obtained by the three methods is shown by the curves in Figs. 9 and 10.

TABLE 3

VALUES OF ADDITIONAL LONGITUDINAL STRESS AT B DUE TO LOAD APPLIED THROUGH CENTROID OF CHANNEL CROSS-SECTION

Channel	Load† P (lb.)	$e = \frac{h}{2}\frac{x_c}{y_c}$ (in.)	m (in.)	$a = 0.6h\sqrt{\frac{I_y}{J}}$ (in.)	(in.)	b (in.)	$I_f = \frac{1}{12}tb^3$ (in.⁴)	$s_2 = \frac{Pma}{h}\frac{\frac{1}{2}b}{I_f}$ (lb. per sq. in.)	Values of s_2 from Tests (lb. per sq. in.)	Values of s_2 from Eqs. 18 and 49 in App. (lb. per sq. in.)
4 in.–5.4 lb.....	550	0.47	0.84	7.6	0.296	1.58	0.0974	7 120	Not tested	4300
6 in.–8.2 lb.....	800	0.56	0.98	12.0	0.343	1.92	0.196	7 660	5 400	5300
6 in.–15.3 lb. (Ship)..	1600	1.05	1.96	18.2	0.385	3.50	1.37	12 100	10 000	8700
6 in.–15.5 lb. (Heavy)	1250	0.42	1.14*	6.48	0.343	2.28	0.339	5 170*	Indefinite	2800*
10 in.–15.3 lb.....	1787	0.70	1.22	21.0	0.436	2.60	0.637	9 380	11 200	8400
15 in.–33.9 lb.....	5560	0.78	1.37	27.6	0.650	3.40	2.13	11 200	11 200	8700



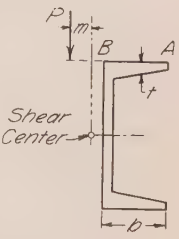
*Load applied 0.45 in. to right of centroid.

†Load in each case gives a value for s in $s = \frac{M_c}{I}$ of 16 000 lb. per sq. in. at outer fibers.

TABLE 4

VALUES OF ADDITIONAL LONGITUDINAL STRESS AT *A* DUE TO LOAD APPLIED BACK OF SHEAR CENTER OF CHANNEL CROSS-SECTION

Channel	P (lb.)	m (in.)	$s_2 = \frac{Pma}{h}\frac{\frac{1}{2}b}{I_f}$ (lb. per sq. in.)	Values of s_2 from Tests (lb. per sq. in.)	Values of s_2 from Eqs. 18 and 49 in App. (lb. per sq. in.)
4 in.–5.4 lb.....	550	1.00	8 500	Not tested	8 700
6 in.–8.2 lb.....	800	1.04	8 150	7 400	9 900
6 in.–15.3 lb. (Ship)...	1600	1.12	6 950	6 400	7 700
6 in.–15.5 lb. (Heavy)..	1250	0.93	4 220	Indefinite	4 700
10 in.–15.3 lb.....	1787	1.00	7 660	Not tested	13 600
15 in.–33.9 lb.....	5560	1.42	11 600	16 200	19 200



For loads applied approximately through the centroids of the channel sections tested the maximum additional longitudinal stress caused by the twisting of the channels varied from approximately 34 per cent of the stress $\left(s = \frac{Mc}{I}\right)$ due to the transverse bending moment in the case of the 6-in. 8.2-lb. channel, to approximately 70 per cent, in the case of the 15-in. 33.9-lb. channel. The 6-in. 15.5-lb. (heavy) channel was not tested with the load applied through the centroid. However, it would no doubt have developed an additional longitudinal stress considerably less than 34 per cent of the stress due to the transverse bending moment, for the torsional stiffness of the section increases rapidly as the thickness of the web increases, and hence the twisting of the section and the accompanying additional longitudinal stress decreases.

12. *Significance of Effect of Twisting of Channel Beams.*—It is clear from the results presented in the preceding sections that the longitudinal stress in a channel when used as a beam subjected to transverse loads that do not pass through the shear center of the section may be greatly in excess of that given by the simple flexure formula, provided that the beam is free to twist as it bends. However, the torsional stiffness of channels is very small, and hence a relatively small lateral restraint at a number of points along the beam is sufficient to prevent the twisting that produces the large additional longitudinal stress at the fixed section of the beam.

In many uses of a channel beam, therefore, even though the loads are not applied through the shear center, the lateral restraint supplied by attached portions of the structure, such as floors, etc., may justify

the assumption that the longitudinal stresses are approximately the same as though the loads were applied through the shear center.

When channel beams, however, are used so that they can twist as they bend, it is very important that they be loaded through the shear center or, if this is impossible, they should be designed to resist longitudinal stresses considerably in excess of those required by the flexural formula, in accordance with the analysis given in Section 10, or with that given in the Appendix. In any case it is important to understand the limitation of the flexure formula as applied to channel beams.

IV. Conclusion

13. *Summary.*—A brief summary of the main facts brought out in this investigation may be stated as follows:

(1) Steel rolled I beams subjected to transverse loads acting through the centroids of the cross-sections bend without twisting, and develop longitudinal (bending) stresses in accordance with the simple flexure formula for beams. It has been widely assumed that rolled steel channel beams when similarly loaded also bend without twisting and develop longitudinal stresses in accordance with the simple flexure formula, but this assumption may be largely in error. Thus a channel beam subjected to transverse loads through the centroids of the sections may not have bending resistance in proportion to its section modulus as is usually assumed.

(2) In order to cause a channel beam to bend without twisting, and to develop a bending resistance in proportion to its section modulus as is assumed in the simple flexure formula, the loads must lie in a plane parallel to the web but at a considerable distance back from the web. The intersection of this plane of bending with the neutral axis of a transverse cross-section of the channel is called the *shear center* for the section.

(3) The distance of the shear center for a channel section from the center line of the web depends only on the dimensions of the section and is given satisfactorily by the simple expression, see Equation (4),

$$e = \frac{h}{2} \cdot \frac{x_c}{y_c}$$

For the channels tested, which ranged in depth from 4 in. to 15 in., the distance of the shear center from the center line of the web varied from 0.35 in. for the 6-in. 15.5-lb. (heavy) channel having a thick web, to 1.10 in. for the 6-in. 15.3-lb. (ship) channel having a wide flange.

(4) Angle or other connections which for convenience are attached to the back of the web of the channel transfer the transverse loads to the channel beam approximately in the plane which causes the beam to bend without twisting, and thus enable the channel to develop approximately its full bending resistance, as expressed by the simple flexure formula.

(5) When the transverse loads on a channel beam are not applied through the shear center of the sections the beam is subjected, at each section, to a twisting moment in addition to a bending moment, and this twisting moment causes additional longitudinal stresses at any section that is restrained from warping, such as the central section of a simple beam loaded symmetrically, or the fixed end of a cantilever beam.

(6) The maximum value of the additional longitudinal stress caused by the twisting moment resulting from placing the transverse load at various distances from the shear center is given satisfactorily, for the channels tested, by the simple approximate method of Section 10 (Fig. 11), leading to the expression

$$s_2 = \frac{M\frac{b}{2}}{I_f} = \frac{Pm}{h}\, a\, \frac{\frac{1}{2}b}{\frac{1}{12}tb^3} = \frac{6\,Pma}{htb^2}$$

(7) The maximum value of the additional longitudinal stress due to the twisting moment resulting from a transverse load applied approximately through the centroid of the channel sections tested occurred in the 15-in. 33.9-lb. channel, and was equal to 70 per cent of the longitudinal stress $\left(s = \frac{Mc}{I}\right)$ caused by the transverse bending moment. The minimum value of the additional longitudinal stress would no doubt have developed in the 6-in. 15.5-lb. (heavy) channel because of its thick web, but this section was not tested with the load applied through the centroid.

(8) The large additional longitudinal stress caused by a relatively small twisting moment is the result of the very small torsional stiffness of the channel beam. It is evident, therefore, that in many cases of channels in which the loads do not pass through the shear centers of the sections, but in which some lateral restraint at a number of points along the beam is provided, by floors, walls, etc., the twisting of the channel may be prevented, and the longitudinal stress may be substantially the same as though the loads were applied through the shear centers. It is evident also that sections with thick webs are especially

desirable in channels that are free to twist as they bend, since the torsional stiffness of a channel section increases rapidly with the thickness of the web (approximately as the cube of the thickness of the web). But, in any case, it is important that the significance of the limitations of the flexure formula for channel beams be well understood.

Appendix

Bars of Narrow Cross-section Subjected to Torsion with Restraint

I. Introduction

1. *Purpose of Appendix.*—The main purpose of this Appendix is to give a method for computing the major longitudinal stresses normal to a transverse section of a bar which is subjected to bending and torsion or to torsion only, and which is restrained in such a way that one cross-section remains fixed without distortion.

2. *Types of Sections.*—A section to which the method is applicable must be narrow, that is, one dimension of the elements which make up the section must be small compared to the other dimension. A number of such sections are shown in Fig. 12. Sections of the type (f) in Fig. 12 will not be treated here since a graphical method particularly useful for such a section has been given by A. Eggenschwyler.*

3. *Loads and Restraints.*—The loads applied on the bar are restricted to transverse forces only. The resultant of the loads may be a single force subjecting the bar to torsion and bending, or a couple causing torsion only. In either case the single force or couple does not exceed the critical tipping or buckling load.†

One cross-section of the bar is constrained to remain fixed, that is, no torsional warping of the section and no rotation of the section with respect to an axis in its plane takes place. Such fixity may result from conditions of symmetry as, for example, at the central section of a simple beam symmetrically loaded, or from the restraints as at the end of a cantilever, etc.

4. *Assumptions.*—The usual assumptions necessary to idealize an actual structural member and fit it for mathematical treatment are made; namely, that the material is homogeneous and isotropic, and that, in addition, the bar is straight and has a constant cross-section.

5. *Method of Analysis.*—The transverse bending forces on a beam may have a torsional moment with respect to a particular longitudinal axis about which the beam will twist as it bends. This axis is called

*Proceedings of the 2nd International Congress for Applied Mechanics, 1926.

†S. Timoshenko, Zeit. für Math. und Physik., 1910. Vol. 58, p. 337. See also Proc. Math. Soc. (London) Vol. 20, 1922, Ser. 2, p. 389.

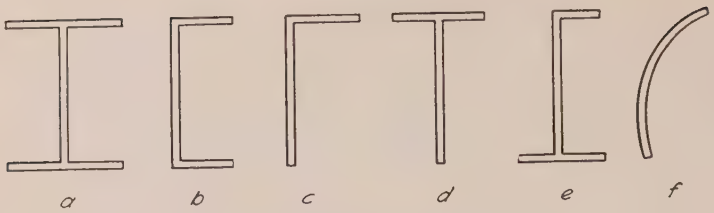


FIG. 12. TYPES OF SECTIONS

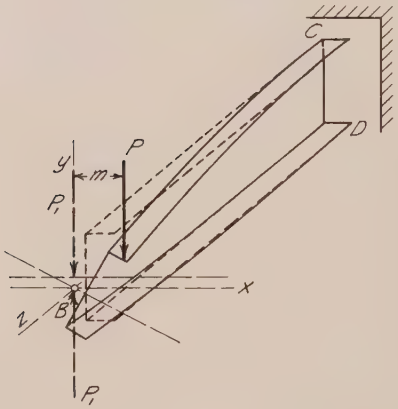


FIG. 13. CHANNEL BEAM SUBJECTED TO TORSION WITH RESTRAINT

the bending axis.* In order that no twist may accompany the bending, the resultant of the transverse loads must intersect the bending axis. The intersection of this axis with the plane of a cross-section of the beam is called the shear center† for the section.

If the resultant of the transverse forces on a bar is a couple, then the bar will rotate with respect to an axis of twist. The intersection of this axis with a cross-section of the bar is called the center of rotation for the section.

A method for determining the longitudinal stresses, whether in a beam under bending and torsion, or in a bar subjected to torsion only, will depend on the location of the shear center or the center of rotation. For example, consider a channel beam, Fig. 13, with a load P, at the free end. The other end is fixed in such a way that the cross-section CD at the support remains plane and does not rotate with respect to an axis in the plane. The load P is replaced by a force $P_1 = P$ through the shear center B, and a twisting couple of moment

*A. Eggenschwyler, "Biegungsaxe," Schweiz. Bauz. Vol. 76, Dec. 4, 1920, p. 266.
†R. Maillart, "Schubmittelpunkt," Schweiz. Bauz., Vol. 77, April 30, 1921, p. 195.

$M = Pm$. The effect of P_1 is a vertical deflection of the bending axis, and the effect of the couple M is a rotation of the cross-section with respect to the shear center as the center of rotation.* The longitudinal stresses which are caused by P_1 alone are obtained from the ordinary flexure formula $s = \frac{M_x y}{I_x}$. For the more general case of bending of a section with no axes of symmetry a more general flexure formula of the form $s = \frac{Mc}{R}$ may be used. In any case, bending caused by a load acting through the bending axis will be referred to as "transverse bending without torsion."

The twisting of a beam or a bar having one section restrained from warping is called "torsion with restraint." For the example in Fig. 13 the longitudinal stresses caused by the torsional couple M will be found, by considering the twisting of the whole beam about its bending axis, to be equivalent to equal and opposite bending in horizontal planes (without torsion) of the upper and lower halves of the beam, combined with twisting of each half about its own axis of bending. In the twisting just referred to, all cross-sections in each half of the beam are free to warp; such twisting is called "torsion without restraint."

Before determining the longitudinal stresses in a beam or bar subjected to torsion with restraint it will be necessary, therefore, to consider the problem of (a) transverse bending without torsion and (b) torsion without restraint.

II. Transverse Bending without Torsion

6. *The Shear Center in General.*—As stated in Section 5, if no twist is to occur in a bar subjected to transverse loads, the plane of the loads must contain the bending axis. For any cross-section of the bar the resultant of the loads and reactions on either side of the section may be replaced by a parallel force of equal magnitude (external shear) through the shear center of the section and a bending couple. The external shear is, of course, balanced by an internal shear. If it is possible to determine the action lines of the internal shear in the section for each of two planes of loading, then the shear center of the section is determined by the intersection of the two action lines of the two internal shears. That two planes of loading are sufficient has

*C. Weber, Zeit. für Angew. Math. und Mech., Vol. 6, 1926, p. 92.

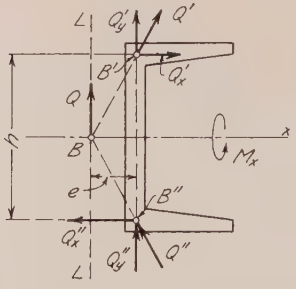


FIG. 14. SECTION OF CHANNEL LOADED THROUGH SHEAR CENTER

been shown by R. Maillart.* If one plane of symmetry exists for the bar, as in the case of a channel section, it is evident that the internal shear will lie in this plane. Hence only one plane of loading other than the plane of symmetry need be considered. If two planes of symmetry exist, as in the case of the I section, the shear center is determined by the intersection of the planes of symmetry with the cross-section.

For the angle section it is evident from the preceding paragraph that the shear center is approximately the intersection of the center lines parallel to the long sides of the two narrow rectangles which make up the section.

7. *The Shear Center for Channel Sections.*—The channel section has one axis of symmetry, hence the action line of the resultant of the system of shearing stresses for only one plane of loading need be found. Let the line LL, Fig. 14, be the intersection of the plane of the loads with the plane of the cross-section. For a particular location of this plane at a distance e from the web center the torsional effect vanishes and only a pure bending remains; the problem then is to determine the distance e.

From equilibrium it follows that the resultant of all the stresses on a section in the beam at a distance z, measured longitudinally from the origin, is a force which balances the resultant of the loads and reactions on either side of the section. This force is replaced by a force Q with action line LL in the plane of the section and a bending couple of moment, M_x. On a section distant $\pm dz$ from the section at z, the force is $\pm(Q + dQ)$ and the couple $\pm(M_x + dM_x)$. Since the section is narrow it is assumed that the resultant of the shearing

*Schweiz. Bauz., Vol. 77, 1921.

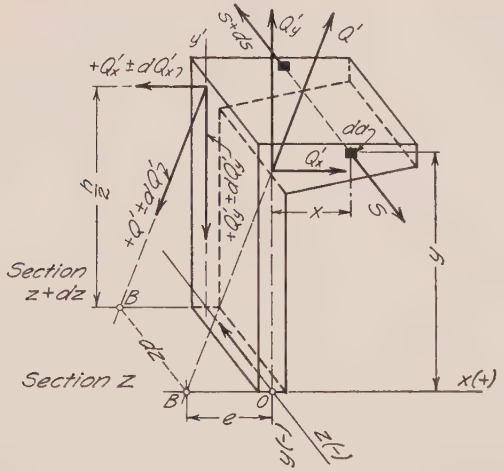


FIG. 15. EQUILIBRIUM OF ELEMENT OF CHANNEL

stress in the section has components which act along the center lines of the web and flange areas. Q may then be resolved into components Q' and Q'', which act through the intersections, B' and B'', of the center line of the web area with the upper and lower flange areas respectively.

Since Q passes through B, the summation of moments of the components of Q with respect to B is equal to zero. Consequently using x and y components of Q' and Q'',

$$Q'_x \frac{h}{2} + Q''_x \frac{h}{2} = (Q'_y + Q''_y)e \tag{1}$$

Since by symmetry

$$Q'_x = Q''_x \text{ and } Q'_y + Q''_y = Q \tag{2}$$

then

$$Qe = Q'_x h \tag{3}$$

In order to determine Q'_x and Q it is necessary to consider the equilibrium of an element of length dz (Fig. 15). In general, four equilibrium combinations of bending moment M_x and shear Q are possible. Figure 15 shows the element in equilibrium when M_x on section at z is negative and Q on section at z is positive. Then on section $z + dz$ the moment is $-M + dM$, and the shear is $+Q \pm dQ$.

It follows that the difference ds in the normal stress, on an element of area dA, has a negative sense in the z direction. Hence the moment of the stresses with respect to the web axis Y' in section $z + dz$ is clockwise when viewed from the upper end of the Y' axis, and this moment must be balanced by a counter clockwise moment $Q'_x dz$. The sense of Q'_x being thus determined as positive means that Q', and therefore Q, acts through the point B at $x = -e$. Any other combination of M_x and Q leads to the same result.

Since the section is in pure bending it is assumed that the normal stresses are proportional to the distance y from the neutral axis X, or,

$$\frac{s}{y} = \frac{M_x}{I_x} \tag{4}$$

From equilibrium the moment of all the stresses acting on the element in Fig. 15 with respect to the Y' axis is equal to zero, or,

$$\int_{A'} x \, ds \, da = Q'_x \, dz \tag{5}$$

where A' is the area of the half-section of the channel. Evaluating the integral in (5)

$$\int_{A'} x \, ds \, da = I'_{xy} \frac{ds}{y} \tag{6}$$

where I'_{xy} is the product of inertia of the half section. Hence,

$$Q'_x = \frac{ds}{dz} \frac{I'_{xy}}{y} \tag{7}$$

From the relation $Q = \dfrac{dM_x}{dz}$ and (4)

$$Q = \frac{ds}{dz} \frac{I_x}{y} \tag{8}$$

Then from (3)
$$e = -\frac{Q'_x h}{Q} = \frac{-h I'_{xy}}{I_x} \tag{9}$$

This formula for e has been derived by A. Eggenschwyler,* C. Weber,† and others.

*Schweiz. Bauz. Vol. 77., April 30, 1921, p. 195.
†Zeit. für Angew. Math. und Mech., Vol. 4, 1924, p. 347.

III. Torsion without Restraint

8. *Torsion Formula.*—If all of the cross-sections in a bar subjected to twisting are free to warp, the generators of the surface of the bar remain practically straight lines with negligible change in their lengths unless the angle of twist is large and the cross-sections are unusually extended. For rolled shapes as commonly used the longitudinal stresses which accompany these changes in length may be neglected.* The relation between a twisting moment M_t and the angle of twist per unit of length, $\frac{d\theta}{dz}$, for a bar of non-circular cross-section, may be written in the same form as that for a circular section,

$$M_t = E_s J \frac{d\theta}{dz} \tag{10}$$

where $E_s J$ is the torsional stiffness, E_s is the shearing modulus of elasticity and J is a modified polar moment of inertia. For a narrow rectangular section of width d, and length l,

$$J = \tfrac{1}{3}\,(l - 0.63d)d^3 \text{ (approx.)} \tag{11}$$

by the St. Venant theory of torsion. If the $0.63d$ is neglected, then

$$J = \tfrac{1}{3}\,ld^3 \text{ (approx.)} \tag{12}$$

For a rolled section which may be regarded as made up of narrow rectangles, A. Föppl† proposed

$$J = \tfrac{1}{3}\sum ld^3 \tag{13}$$

From tests on rolled sections, A. Föppl‡ found that a correction factor which varied from 0.86 to 1.47 was necessary to apply to Equation (13) in order to fit the test results. C. Weber§ found by analysis that

$$J = 1.06 \cdot \tfrac{1}{3}\sum ld^3 \tag{14}$$

and from tests

$$J = 1.09 \cdot \tfrac{1}{3}\sum ld^3 \tag{15}$$

Table 5 shows values of J as computed from Equation (13) and as determined from the tests of the channels reported on in the main body of this bulletin, the average value of the correction factor being 1.10.

*C. Weber, Forschungsarbeiten, Heft., 249, 1921, p. 61.
†A. Föppl, K. B. Akad. der Wissenschaften, 1917, p. 5.
‡*Ibid.*, 1921, p. 295.
§C. Weber, A. Föppl, Festschrift, p. 104.

TABLE 5

VALUES OF J FOR VARIOUS CHANNEL SECTIONS

Channel Section in. lb.	$\frac{1}{3}\Sigma ld^3$* in.^{4}	J Test		Correction Factor k in $J = k\ \frac{1}{3}\Sigma ld^3$	
		Torsion† Only in.^{4}	Combined‡ Bending and Torsion in.^{4}	Ratio 3/2	Ratio 4/2
15–33.9	0.890	1.080	0.897	1.21	1.01
10–15.3	0.179	0.255	0.188	1.42	1.05
6–15.5	0.352	0.393	0.356	1.11	1.01
6–15.3 (Ship)	0.206	0.208	0.222	1.02	1.08
6–8.2	0.059	0.072	0.074	1.22	1.25
4–5.4	0.034	.....	0.040	....	1.18
					Av. 1.10

*Measured dimensions were used in computing these values.
†The tests on the channels in torsion only are not reported in detail.
‡Values obtained by use of the straight line portion of curves in Fig. 5.

IV. TORSION WITH RESTRAINT

9. *Torsion with Restraint Equivalent to Bending Plus Torsion without Restraint.*—If one cross-section of a channel bar under the action of a twisting couple M, only, is restrained from warping, then the generators of the surface become decidedly curved with marked changes in their lengths, and the accompanying longitudinal stresses are not negligible. This curvature of the generators implies a state of bending moment, which is of opposite sign in the upper and lower halves of the bar. Just as the action of P, for example, on the whole beam (Fig. 13) may be resolved into a bending action plus a twisting action (with restraint) about the bending axis of the whole beam, so the twisting action of the whole beam may again be resolved into opposite bending actions of the upper and lower halves of the bar plus a twisting without restraint, characterized by Equation (10), of the upper and lower halves about their respective axes of rotation, provided that in the bending of each half certain conditions are fulfilled by the neutral axes of the half cross-sections.

Let a twisting moment M, only, (Fig. 16a) be applied at the free end of a channel bar, the other end remaining plane and fixed. Consider a section at a distance z from the free end. The rotation of this section with respect to the fixed section is shown in Fig. 16b. The upper half section is considered to deflect in bending through a dis-

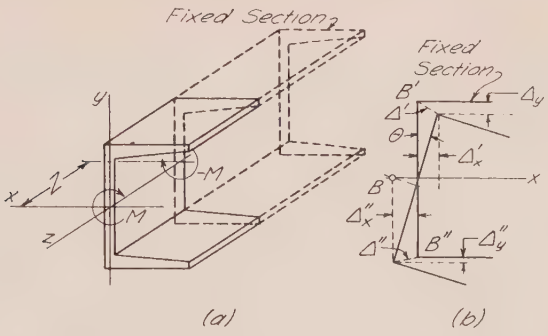


FIG. 16. ROTATION OF CHANNEL SUBJECTED TO TORSION

tance Δ' to the dotted position, then to twist, through the angle θ, about its center of rotation B' (in the new position) to its final position, where it joins with the web of the lower half which has proceeded through similar deformations in the opposite sense. The bending and twisting proceed, of course, simultaneously, so that continuity is never broken. In order to formulate the resolution of the action as described it is necessary to know the location of the center of rotation B of the whole section, as well as the centers of rotation B' and B'' for the half (angle) sections. It is possible to locate these centers from the conditions which must be satisfied in the method just outlined, or from the results in Sections 6 and 7.

10. *Center of Rotation for an Angle Section.*—As a simple example of the method proposed in the preceding section, consider the upper half of the channel bar as an angle bar fixed at one end and under the action of a twisting couple which rotates the section (Fig. 17, not necessarily a very narrow section) through a small angle θ, (Fig. 17b), about some center B, with coördinates x_B, y_B.

The rotation of the whole angle I-II, about B' may be resolved into a bending deflection of I to I′ plus a rotation of I about B_1, and a bending deflection of II to II′ plus a rotation of II about B_2. In order to determine x_B and y_B, the following conditions as to equilibrium and continuity, and assumptions as to stress must be satisfied:

(1) The bending moment on the whole angle section is zero since the angle is subjected to a twisting moment only. Hence the bending moments in I and II are equal and of opposite sign.

(2) The section remains continuous at B'.

(3) The normal stress at B' due to bending in I, is equal to the normal or bending stress at B' in II.

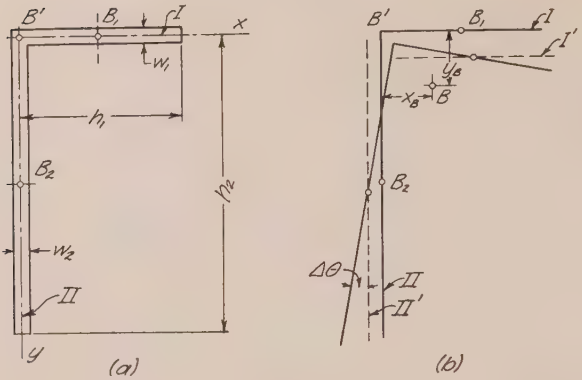


FIG. 17. ROTATION OF ANGLE SUBJECTED TO TORSION

(4) Normal or bending stresses in I and II are proportional to distances from the neutral axes.

Since the bending deflections of I and II are perpendicular to their neutral axes, it is evident that the center of rotation B' for a small angle of rotation is the intersection of the neutral axes of I and II. For the angle section it can be shown by use of the conditions stated that the coördinates for the center of rotation B are approximately

$$x_B = +\tfrac{1}{2}\frac{w^3{}_2}{w_1}\frac{h_2}{h_1{}^2}$$

$$y_B = -\tfrac{1}{2}\frac{w^3{}_1}{w_2}\frac{h_1}{h_2{}^2} \qquad (16)$$

If the width w of the sections becomes very small then w^3 approaches zero and x_B and y_B become approximately equal to zero, that is B' becomes the center of twist. This agrees with the location of the shear center given in Section 6. If $h_1 = 0$, that is, if the angle section degenerates into a rectangular section, then $x_B = \dfrac{h_2}{2}$ and $y_B = 0$.

11. *Center of Rotation for a Channel Section.*—If the method suggested in Section 10 for the angle section is applied to the channel section, Fig. 18, the center of rotation B is found as the intersection of lines drawn parallel to the neutral axes N_1 and N_2 for the bending of the half sections, through the points B_1 and B_2 respectively. On account of the symmetry of this section, and since the bending mo-

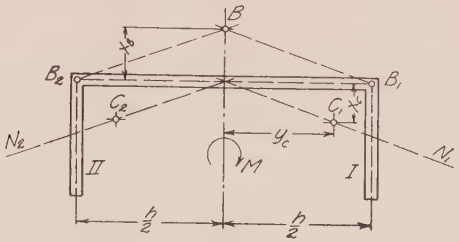


FIG. 18. CENTER OF ROTATION FOR CHANNEL SECTION

ments in the half sections are of opposite sign, the point of zero bending stress is the web center. Consequently the neutral axis of the half section is determined by the web center and the centroid of the half section. The distance from the web center to B is then

$$x_B = \frac{h}{2}\,\frac{x_c}{y_c} \tag{17}$$

where x_c, y_c are the coördinates of the centroid of the half section.

12. *Longitudinal Stress Formula.*—As stated in Section 9, a part of the effect of the torsional couple acting on the channel may be regarded as a bending which is unsymmetrical, and hence the stress formula from the theory of unsymmetrical bending is used. For the upper half of the beam (Fig. 16a) this formula is

$$s = \frac{M'c'}{R'} \tag{18}$$

where c' is the distance from the neutral axis of the upper half to the point at which the stress s exists, R' is the "directed" moment of inertia* of the upper half of the cross-section, and M' is the bending moment in the upper half of the cross-section. The derivation for Equation (18) as well as a graphical method for the determination of R' follows.

Consider any transverse section (Fig. 19) of a beam which bends with respect to any neutral axis Y' that makes an angle θ with the Y axis. The moment of the normal stress with respect to the Y' axis is

$$M_{y'} = \int s\,x'\,da \tag{19}$$

*H. M. Westergaard, Lectures in Advanced Tech. Mech. at Univ. of Ill.

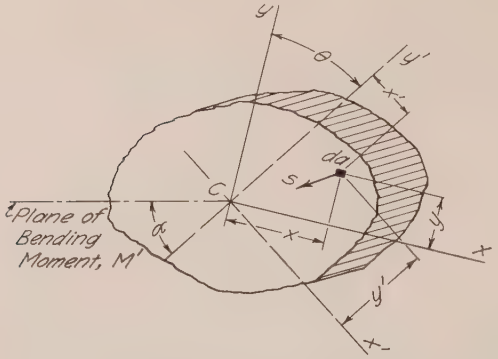


FIG. 19. SECTION OF BEAM SUBJECTED TO UNSYMMETRICAL BENDING

Assuming that $\frac{s}{x'}$ is constant

$$M_{y'} = \frac{s}{x'} I_{y'} \tag{20}$$

where $I_{y'} = \int (x')^2 \, da$

Similarly

$$M_{x'} = \int s \, y' \, da = \frac{s}{x'} I_{x'y'} \tag{21}$$

where $$I_{x'y'} = \int x'y'da$$

Then $$s = \frac{M'x'}{\sqrt{I^2_{y'} + I^2_{x'y'}}} = \frac{M'x'}{R'} \tag{22}$$

where $R' = \sqrt{I^2_{y'} + I^2_{x'y'}}$ and $M' = \sqrt{M^2_{x'} + M^2_{y'}}$

The direction of the plane of the bending moment M' referred to the neutral axis Y' is given by the angle α whose tangent is

$$\tan \alpha = \frac{M_{y'}}{M_{x'}} = \frac{I_{y'}}{I_{x'y'}} \tag{23}$$

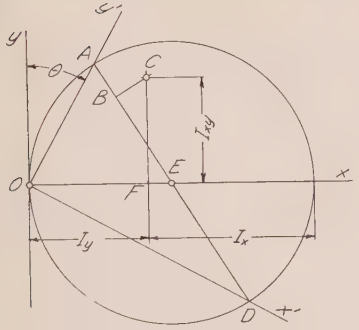


Fig. 20. Circle of Inertia

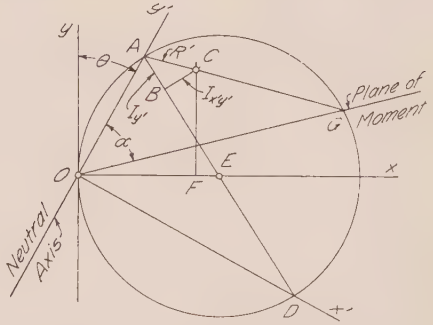


Fig. 21. Circle of Inertia Applied to Unsymmetrical Bending

For axes x and y the directed moment of inertia is $R' =$

$$\sqrt{\tfrac{1}{2}(I_x^2 + I_y^2 + 2I_{xy}) - \tfrac{1}{2}(I_x^2 - I_y^2)\cos 2\theta - (I_x I_{xy} + I_y I_{xy})\sin 2\theta} \quad (24)$$

The quantity R' is conveniently found by the following graphical construction, which is based on methods developed by O. Mohr and R. Land.*

I_y and I_x are scaled off from the origin O and F, respectively, (Fig. 20) on the X axis, and I_{xy}, the product of inertia, is measured from F in the Y direction, upwards if positive, and downwards if negative. A circle with center E, where $OE = \dfrac{I_y + I_x}{2}$, is then drawn. Then OY' is drawn at any angle θ with the Y axis and a diameter AD is drawn from A, the intersection of OY' with the circle. A perpendicular from C is next drawn to the diameter AD. It can be shown that $I_{y'}$ is given by the length AB, $I_{x'}$ by BD, and $I_{x'y'}$ by BC. $I_{x'y'}$ is positive if B is on the left of a line through C and E, and negative if on the right of the line.

Referring to Fig. 21, if the Y' line represents the neutral axis for an unsymmetrical bending, and a line is drawn from A through C to G on the circle, then AC measures the quantity R' and the direction OG represents the direction of the plane of the bending moment M'.

13. *Bending Shear Couple.*—From the equilibrium of a length z of the bar (Fig. 16a) the applied twisting couple M at the end is balanced by the stress couple $-M$. The stress couple $-M$, according to the

*L. J. Johnson in Jour. Assoc. Eng. Soc., May 1902, p. 257.

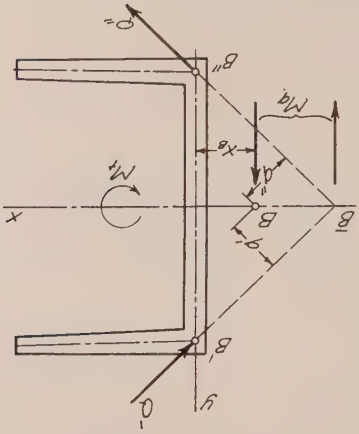


FIG. 22. BENDING SHEAR COUPLE

method of analysis in Section 9, is composed of two couples, a torsional couple M_t (torsion without restraint), and a bending shear couple M_q (Fig. 22). Since M' is the bending moment in the upper half section, then

$$\frac{dM'}{dz} = Q' . \tag{24}$$

where Q' is the bending shear in the upper half section. Similarly, Q'' is the bending shear in the lower half section. Q' and Q'' are the two components of one of the forces of the couple M_q, through the point $\bar{B}$. The other force of the couple acts through B. Q' and Q'' must act through the shear centers B' and B'' of the half sections, since the bending part of the action is without torsion. The bending shear couple may then be written as

$$M_q = Q'q' + Q''q'' \tag{25}$$

Since from symmetry $Q' = Q''$ and $q' = q''$

$$M_q = 2Q'q' \tag{26}$$

14. *Differential Equation for Torsion with Restraint.*—From the equilibrium of an element of length z of the beam

$$M = M_t + M_q \tag{27}$$

Substituting $M_t = E_s J \frac{d\theta}{dz}$ and $M_q = 2Q'q'$, Equation (27) becomes

$$M = 2Q'q' - E_s J \frac{d\theta}{dz} \tag{28}$$

With the origin at the free end, $\frac{d\theta}{dz}$ is a negative quantity.

Since M' represents the bending moment in the upper half section (due to curvature), then, in terms of the deflection, the bending moment becomes

$$M' = E R' \frac{d^2\Delta'}{dz^2} \tag{29}$$

and

$$Q' = \frac{dM'}{dz} = E R' \frac{d^3\Delta'}{dz^3} \tag{30}$$

Δ', the deflection of the center of rotation (shear center) of the upper half section at distance z from the origin, may be written in terms of θ, the angle of twist.

$$\Delta' = \sqrt{(\Delta'_x)^2 + (\Delta'_y)^2} = \sqrt{\frac{h^2}{4} + x^2{}_B} \cdot \theta \tag{31}$$

then

$$M = 2q'ER'\sqrt{\frac{h^2}{4} + x^2{}_B}\,\frac{d^3\theta}{dz^3} - E_s J \frac{d\theta}{dz} \tag{32}$$

Let

$$C = E_s J \tag{33}$$

$$D = 2q'\,ER'\sqrt{\frac{h^2}{4} + x^2{}_B} \tag{34}$$

$$a^2 = \frac{D}{C} \tag{35}$$

then (32) becomes

$$\frac{d^3\theta}{dz^3} - \frac{l}{a^2}\frac{d\theta}{dz} = \frac{M}{a^2 C} \tag{36}$$

*S. Timoshenko, Zeit. für Math. und Physik, Vol. 58, 1910.
L. Föppl and K. Huber, Der Bauingenieur, Vol. 6, 1925, p. 182.

The solution of this equation is

$$\theta = -A_0 + A_1\theta^{\frac{z}{a}} + A_2 e^{\frac{-z}{a}} - \frac{Mz}{C} \tag{37}$$

The constants of integration are found from the boundary conditions which are

$$\begin{aligned} &\text{for } z = 0, \qquad M_q = 0 \qquad \frac{d^2\theta}{dz^2} = 0 \\ &z = l, \qquad \theta = 0 \qquad \frac{d\theta}{dz} = 0 \end{aligned} \tag{38}$$

Hence,

$$A_1 = -A_2;\ A_2 = -\frac{M}{C}\frac{a}{e^{l/a} + e^{-l/a}};\ -A_0 = \frac{M}{C}\left(l - a\,\frac{e^{l/a} - e^{-l/a}}{e^{l/a} + e^{-l/a}}\right)$$

Then

$$\theta = \frac{M}{C}\left[l - a\tanh\frac{l}{a} + a\,\frac{\sinh\frac{z}{a}}{\cosh\frac{l}{a}} - z\right] \tag{39}$$

θ is a maximum (denoted by θ_0) when $z = 0$. Thus,

$$\text{Maximum } \theta = \theta_0 = \frac{M}{C}\left[l - a\tanh\frac{l}{a}\right] \tag{40}$$

Since $\tanh\frac{a}{l}$ is very nearly equal to unity for all of the cases considered

$$\theta_0 = \frac{M}{C}(l - a) \tag{41}$$

15. *Variation of Torsional Couple and Bending Shear Couple along Length of Bar.*—From Equations (26), (30), and (31),

$$M_{q'} = 2ER'q'\sqrt{\frac{h^2}{4} + x^2{}_B}\,\frac{d^3\theta}{dz^3} = D\,\frac{d^3\theta}{dz^3} \tag{42}$$

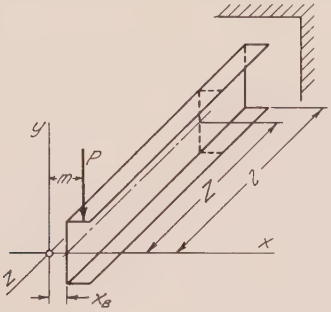


FIG. 23. CHANNEL BEAM WITH LOAD NOT THROUGH SHEAR CENTER

But from Equation (39)

$$\frac{d^3\theta}{dz^3} = \frac{M}{a^2C} \frac{\cosh \frac{z}{a}}{\cosh \frac{l}{a}} \tag{43}$$

hence the bending shear couple is

$$M_q = M \frac{\cosh \frac{z}{a}}{\cosh \frac{l}{a}} \tag{44}$$

When $z = l$, $M_q = M$ (at fixed end).

The torsional couple is

$$M_t = -C \frac{d\theta}{dz} = -M \left[\frac{\cosh \frac{z}{a}}{\cosh \frac{l}{a}} - 1 \right] \tag{45}$$

For $z = l$, $M_t = 0$ (at fixed end).

16. *Bending Moment M' in the Half Section.*—Since

$$M_q = 2Q'q' = M \frac{\cosh \frac{z}{a}}{\cosh \frac{l}{a}} \tag{46}$$

$$M' = \int_0^z Q'dz = \frac{M}{2q'}\, a\, \frac{\sinh \frac{z}{a}}{\cosh \frac{l}{a}} \tag{47}$$

$$\text{for } z = l,\ M' = M' \text{ (maximum)} = \frac{M}{2q'}\, a \tanh \frac{l}{a} \tag{48}$$

Putting $\tanh \frac{l}{a}$ equal to unity as before, the maximum value of M' becomes

$$M'_{max} = \frac{Ma}{2q'} \tag{49}$$

17. *Computation of Longitudinal Stresses in a 6-in. 15.3-lb. Ship Channel.*—The properties of a 6-in. 15.3-lb. Carnegie C 109 ship channel are as follows:

$h = 5.61$ in.
$x_c = 0.91$ in.
$y_c = 2.22$ in.
$x_B = 1.15$ in.

For Whole Section

J (Equation 15a) $= 0.218$ in.^{4}
$I_x = 25.3$ in.^{4}

For Half Section

$I_x = 1.73$ in.^{4}
$I_y = 2.55$ in.^{4}
$I_{xy} = 1.20$ in.^{4}
$I_{y'}$ (Equation 51) $= 1.59$ in.^{4}
$I_{y'}$ (Fig. 24) $= 1.60$ in.^{4}
R' (Equation 24) $= 1.90$ in.^{4}
R' (Fig. 24) $= 1.95$ in.^{4}
q' (Equation 50) $= 2.52$ in.
q' (Fig. 24) $= 2.47$ in.
a (Equation 35a) $= 18.5$ in.
$l = 84$ in.

The channel is loaded as shown in Fig. 23. The longitudinal stress at any point in a cross-section z inches from the origin is

$$s = s_1 + s_2$$

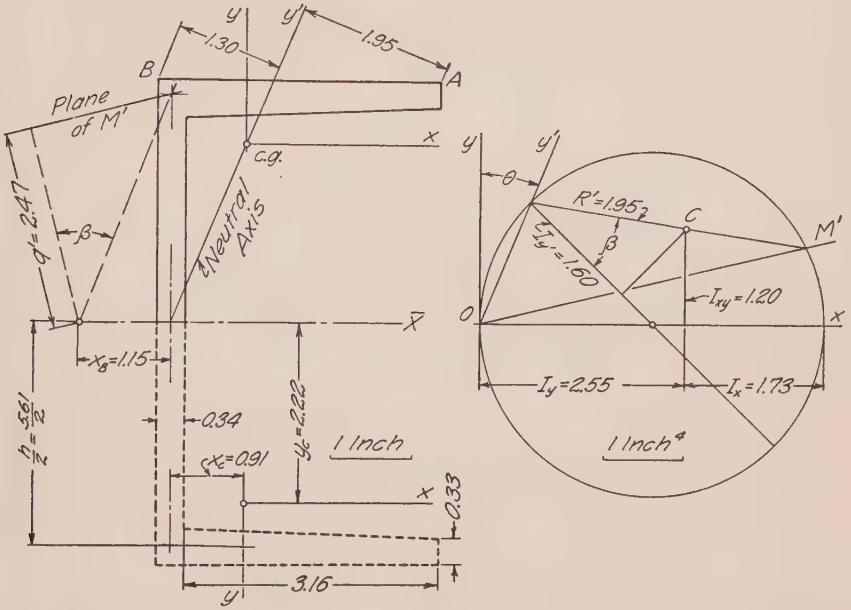


FIG. 24. GRAPHICAL DETERMINATION OF SECTION FACTORS FOR A SHIP CHANNEL

where $$s_1 = \frac{M_{\bar{x}} Y}{I_{\bar{x}}}$$

and $$s_2 = \frac{M'c'}{R'} \tag{18}$$

The bending moments are $$M_{\bar{x}} = Pz$$

and $$M' = \frac{Ma}{2q'} \frac{\sinh \frac{z}{a}}{\cosh \frac{l}{a}} \tag{47}$$

From (33), (34), (35) $$a^2 = 2q' \frac{ER'}{E_s J} \sqrt{\frac{h^2}{4} + x^2{}_B} \tag{35a}$$

From Fig. 24 $$q' = \frac{I_{y'}}{R'} \sqrt{\frac{h^2}{4} + x^2{}_B} \tag{50}$$

From Fig. 21 or 24 $$I_{y'} = \frac{1}{\frac{h^2}{2} + 2x^2{}_B}\left(I_y\frac{h^2}{2} + 2x^2{}_B I_x - 2hx_B I_{xy}\right) \quad (51)$$

From Table 5 $$J = 1.10 \cdot \tfrac{1}{3}\sum ld^3 \quad (15a)$$

For $E = 30\,000\,000$ and $E_s = 12\,000\,000$ (lb. per sq. in.)

from (35a) $$a = 1.12h\sqrt{\frac{I_{y'}}{J}\left(1 + \frac{4x^2{}_B}{h^2}\right)} \quad (35b)$$

Since $$\theta_0 = \frac{M}{E_s J}(l - a) \quad (41)$$

then $$m = x_B + x_c + \theta_0\frac{6}{2} = 2.34 \text{ in.}$$

For $P = 1600$ lb. then from (47) $$M' = \frac{3740 \times 18.5}{2 \times 2.47}\,\frac{\sinh\frac{z}{a}}{\cosh\frac{l}{a}}$$

$$\cosh\frac{l}{a} = \cosh\frac{84}{18.5} = 46.85$$

$$s_1 = \frac{M_{\bar{x}}y}{I_{\bar{x}}} = \frac{1600 \times z \times 3}{25.3} = 189.5\,z$$

$$s_{2A} = \frac{M'c'}{R'} = \frac{1.95}{1.95}M', \quad s_{2B} = \frac{1.30}{1.95}M' = 0.667M'$$

$$s_A = +s_{1A} - s_{2A}, \quad s_B = +s_{1B} + s_{2B}$$

TABLE 6

LONGITUDINAL STRESSES IN 6-IN. 15.3-LB. SHIP CHANNEL

z	$\frac{z}{a}$	$\frac{\text{Sinh } \frac{z}{a}}{\text{Sinh } \frac{l}{a}}$	M'	s_1		s_2		s	
				A (+)	B (+)	A (−)	B (+)	A (+)	B (+)
in.			in.-lb.	lb. per sq. in.					
84	4.54	46.84	1.000	14 000	15 900	14 000	9350	1900	25 250
74	4.00	27.30	0.582	8 150	14 000	8 150	5450	5850	19 450
64	3.46	15.89	0.338	4 750	12 100	4 750	3150	7350	15 250
54	2.92	9.24	0.197	2 750	10 200	2 750	1850	7450	12 050
44	2.16	4.28	0.092	1 300	8 350	1 300	850	7050	9 200
34	1.84	3.07	0.066	950	6 450	950	650	5500	7 100

(+) Tension
(−) Compression

RECENT PUBLICATIONS OF THE ENGINEERING EXPERIMENT STATION†

Bulletin No. 167. Freight Train Curve-Resistance on a One-Degree Curve and a Three-Degree Curve, by Edward C. Schmidt. 1927. *Twenty-five cents.*

Bulletin No. 168. Heat Transmission Through Boiler Tubes, by Huber O. Croft. 1927. *Thirty cents.*

Bulletin No. 169. Effect of Enclosures on Direct Steam Radiator Performance, by Maurice K. Fahnestock. 1927. *Twenty cents.*

Bulletin No. 170. The Measurement of Air Quantities and Energy Losses in Mine Entries. Part II, by Alfred C. Callen and Cloyde M. Smith. 1927. *Forty-five cents.*

Bulletin No. 171. Heat Transfer in Ammonia Condensers, by Alonzo P. Kratz, Horace J. Macintire, and Richard E. Gould. 1927. *Thirty-five cents.*

Bulletin No. 172. The Absorption of Sound by Materials, by Floyd R. Watson. 1927. *Twenty cents.*

Bulletin No. 173. The Surface Tension of Molten Metals, by Earl E. Libman. 1927. *Thirty cents.*

**Circular No. 16.* A Simple Method of Determining Stress in Curved Flexural Members, by Benjamin J. Wilson and John F. Quereau. 1927. *Fifteen cents.*

Bulletin No. 174. The Effect of Climatic Changes upon a Multiple-Span Reinforced Concrete Arch Bridge, by Wilbur M. Wilson. 1927. *Forty cents.*

Bulletin No. 175. An Investigation of Web Stresses in Reinforced Concrete Beams. Part II. Restrained Beams, by Frank E. Richart and Louis J. Larson. 1928. *Forty-five cents.*

Bulletin No. 176. A Metallographic Study of the Path of Fatigue Failure in Copper, by Herbert F. Moore and Frank C. Howard. 1928. *Twenty cents.*

Bulletin No. 177. Embrittlement of Boiler Plate, by Samuel W. Parr and Frederick G. Straub. 1928. *None Available.*

Bulletin No. 178. Tests on the Hydraulics and Pneumatics of House Plumbing. Part II, by Harold E. Babbitt. 1928. *Thirty-five cents.*

Bulletin No. 179. An Investigation of Checkerbrick for Carbureters of Water-gas Machines, by C. W. Parmelee, A. E. R. Westman, and W. H. Pfeiffer. 1928. *Fifty cents.*

Bulletin No. 180. The Classification of Coal, by Samuel W. Parr. 1928. *Thirty-five cents.*

Bulletin No. 181. The Thermal Expansion of Fireclay Bricks, by Albert E. R. Westman. 1928. *Twenty cents.*

Bulletin No. 182. Flow of Brine in Pipes, by Richard E. Gould and Marion I. Levy. 1928. *Fifteen cents.*

Circular No. 17. A Laboratory Furnace for Testing Resistance of Firebrick to Slag Erosion, by Ralph K. Hursh and Chester E. Grigsby. 1928. *Fifteen cents.*

Bulletin No. 183. Tests of the Fatigue Strength of Steam Turbine Blade Shapes, by Herbert F. Moore, Stuart W. Lyon, and Norville J. Alleman. 1928. *Twenty-five cents.*

Bulletin No. 184. The Measurement of Air Quantities and Energy Losses in Mine Entries. Part III, by Alfred C. Callen and Cloyde M. Smith. 1928. *Thirty-five cents.*

Bulletin No. 185. A Study of the Failure of Concrete Under Combined Compressive Stresses, by Frank E. Richart, Anton Brandtzaeg, and Rex L. Brown. 1928. *Fifty-five cents.*

**Bulletin No. 186.* Heat Transfer in Ammonia Condensers. Part II, by Alonzo P. Kratz, Horace J. Macintire, and Richard E. Gould. 1928. *Twenty cents.*

**Bulletin No. 187.* The Surface Tension of Molten Metals. Part II, by Earl E. Libman. 1928. *Fifteen cents.*

---

†Copies of the complete list of publications can be obtained without charge by addressing the Engineering Experiment station, Urbana, Ill.

*A limited number of copies of the bulletins starred are available for free distribution.

Bulletin No. 188. Investigation of Warm-air Furnaces and Heating Systems. Part III, by Arthur C. Willard, Alonzo P. Kratz, and Vincent S. Day. 1928. *Forty-five cents.*

Bulletin No. 189. Investigation of Warm-air Furnaces and Heating Systems. Part IV, by Arthur C. Willard, Alonzo P. Kratz, and Vincent S. Day. 1929. *Sixty cents.*

**Bulletin No. 190.* The Failure of Plain and Spirally Reinforced Concrete in Compression, by Frank E. Richart, Anton Brandtzaeg, and Rex L. Brown. 1929. *Forty cents.*

Bulletin No. 191. Rolling Tests of Plates, by Wilbur M. Wilson. 1929. *Thirty cents.*

Bulletin No. 192. Investigation of Heating Rooms with Direct Steam Radiators Equipped with Enclosures and Shields, by Arthur C. Willard, Alonzo P. Kratz, Maurice K. Fahnestock, and Seichi Konzo. 1929. *Forty cents.*

Bulletin No. 193. An X-Ray Study of Firebrick, by Albert E. R. Westman. 1929. *Fifteen cents.*

**Bulletin No. 194.* Tuning of Oscillating Circuits by Plate Current Variations, by J. Tykocinski-Tykociner and Ralph W. Armstrong. 1929. *Twenty-five cents.*

Bulletin No. 195. The Plaster-Model Method of Determining Stresses Applied to Curved Beams, by Fred B. Seely and Richard V. James. 1929. *Twenty cents.*

**Bulletin No. 196.* An Investigation of the Friability of Different Coals, by Cloyde M. Smith. 1929. *Thirty cents.*

**Circular No. 18.* The Construction, Rehabilitation, and Maintenance of Gravel Roads Suitable for Moderate Traffic, by Carroll C. Wiley. 1929. *Thirty cents.*

**Bulletin No. 197.* A Study of Fatigue Cracks in Car Axles. Part II, by Herbert F. Moore, Stuart W. Lyon, and Norville J. Alleman. 1929. *Twenty cents.*

**Bulletin No. 198.* Results of Tests on Sewage Treatment, by Harold E. Babbitt and Harry E. Schlenz. 1929. *Fifty-five cents.*

**Bulletin No. 199.* The Measurement of Air Quantities and Energy Losses in Mine Entries. Part IV, by Cloyde M. Smith. 1929. *Thirty cents.*

**Bulletin No. 200.* Investigation of Endurance of Bond Strength of Various Clays in Molding Sand, by Carl H. Casberg and William H. Spencer. 1929. *Fifteen cents.*

**Circular No. 19.* Equipment for Gas-Liquid Reactions, by Donald B. Keyes. 1929. *Ten cents.*

Bulletin No. 201. Acid Resisting Cover Enamels for Sheet Iron, by Andrew I. Andrews. 1929. *Twenty-five cents.*

Bulletin No. 202. Laboratory Tests of Reinforced Concrete Arch Ribs, by Wilbur M. Wilson. 1929. *Fifty-five cents.*

**Bulletin No. 203.* Dependability of the Theory of Concrete Arches, by Hardy Cross. 1929. *Twenty cents.*

**Bulletin No. 204.* The Hydroxylation of Double Bonds, by Sherlock Swann, Jr. 1930. *Ten cents.*

**Bulletin No. 205.* A Study of the Ikeda (Electrical Resistance) Short-Time Test for Fatigue Strength of Metals, by Herbert F. Moore and Seichi Konzo. 1930. *Twenty cents.*

**Bulletin No. 206.* Studies in the Electrodeposition of Metals, by Donald B. Keyes and Sherlock Swann, Jr. 1930. *Ten cents.*

**Bulletin No. 207.* The Flow of Air Through Circular Orifices with Rounded Approach, by Joseph A. Polson, Joseph G. Lowther, and Benjamin J. Wilson. 1930. *Thirty cents.*

**Circular No. 20.* An Electrical Method for the Determination of the Dew-Point of Flue Gases, by Henry Fraser Johnstone. 1929. *Fifteen cents.*

Bulletin No. 208. A Study of Slip Lines, Strain Lines, and Cracks in Metals Under Repeated Stress, by Herbert F. Moore and Tibor Ver. 1930. *Thirty-five cents.*

**Bulletin No. 209.* Heat Transfer in Ammonia Condensers. Part III, by Alonzo P. Kratz, Horace J. Macintire, and Richard E. Gould. 1930. *Thirty-five cents.*

Bulletin No. 210. Tension Tests of Rivets, by Wilbur M. Wilson and William A. Oliver. 1930. *Twenty-five cents.*

**Bulletin No. 211.* The Torsional Effect of Transverse Bending Loads on Channel Beams, by Fred B. Seely, William J. Putnam, and William L. Schwalbe. 1930. *Thirty-five cents.*

---

*A limited number of copies of the bulletins starred are available for free distribution.